DIXMIER GROUPS AND BOREL SUBGROUPS

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1. Introduction and statement of results

It is well known that many interesting algebraic groups, including classical infinite families of simple groups, arise as the automorphism groups of finite-dimensional simple algebras (see, e.g., [GP], [KMRT] and references therein). In this paper, we will examine an infinite-dimensional example of this phenomenon. We will study a family of ind-algebraic groups associated with algebras Morita equivalent to the Weyl algebra $A_1(\mathbb{C})$. Recall that $A_1(\mathbb{C})$ is a simple associative $\mathbb{C}$-algebra isomorphic to the ring of differential operators on the affine line $\mathbb{C}^1$. The algebras Morita equivalent to A_1 can be divided into two separate classes: the matrix algebras over A_1 and the rings $D(X)$ of differential operators on the rational singular curves X with normalization $\tilde{X} \cong \mathbb{C}^1$ (see [SS]). The matrix algebras $\mathcal{M}_k(A_1)$ are classified, up to isomorphism, by their index (the dimension of matrices). A remarkable and much less obvious fact[1] is that the rings $D(X)$ are also classified, up to isomorphism, by a single non-negative integer, which is called the differential genus of X (see [BW2]). In the present paper, we will focus on the automorphism groups of $D(X)$: for each $n \geq 0$, we choose a curve X_n of differential genus n, with $X_0 = \mathbb{C}^1$, and write G_n for the corresponding automorphism group $\mathrm{Aut}_{\mathbb{C}}\, D(X_n)$. The group G_0 is thus the automorphism group of A_1 originally studied by J. Dixmier [D]. We therefore call $\{G_n\}$ the *Dixmier groups*. A theorem of Makar-Limanov [ML2] asserts that G_0 is isomorphic to the group G of symplectic (unimodular) automorphisms of the free associative algebra $R = \mathbb{C}\langle x, y\rangle$, the isomorphism $G \xrightarrow{\sim} G_0$ being induced by the natural projection $R \twoheadrightarrow A_1$. We will use this isomorphism to identify G_0 with G; the groups G_n for $n \geq 1$ can then be naturally identified with subgroups of G. To explain this in more detail we introduce our main characters: the Calogero-Moser varieties

$$\mathcal{C}_n := \{(X, Y) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \,:\, \mathrm{rk}([X, Y] + I_n) = 1\}/\!/\mathrm{PGL}_n(\mathbb{C}) \ .$$

Named after a class of integrable systems in classical mechanics (see [KKS]) these algebraic varieties play an important role in several areas, especially in geometry and representation theory (see, e.g., [N], [EG], [E], [Go] and references therein). They were studied in detail in [W], where it was shown (among other things) that the $\mathcal{C}_n$ are smooth affine irreducible complex symplectic varieties of dimension $2n$. Furthermore, in [BW], it was shown that each $\mathcal{C}_n$ carries a *transitive* G-action, which is obtained, roughly speaking, by thinking of $\mathcal{C}_n$ as a subvariety of n-dimensional representations of R (see Section 2 for a precise definition). It turns out that G_n is isomorphic to the stabilizer of a point for this transitive action: thus, fixing a

[1] This fact was first established in [K] following an earlier work of G. Letzter and L. Makar-Limanov (see [LM, Le]). It was rediscovered independently by G. Wilson and the first author in [BW1]. A conceptual proof and explanations can be found in the survey paper [BW2].

basepoint in $\mathcal{C}_n$, we can identify G_n with a specific subgroup of G. Our general strategy will be to study G_n in geometric terms, using the action of G on $\mathcal{C}_n$.

Main results. Recall that one of the main theorems of [BW] asserts that the action of G on the varieties $\mathcal{C}_n$ is transitive for all n. We extend this result in two ways.

Theorem 1. *For each $n \geq 1$, the action of G on $\mathcal{C}_n$ is doubly transitive.*

Theorem 2. *For any pairwise distinct natural numbers $(n_1, n_2, \ldots, n_m) \in \mathbb{N}^m$, the diagonal action of G on $\mathcal{C}_{n_1} \times \mathcal{C}_{n_2} \times \ldots \times \mathcal{C}_{n_m}$ is transitive.*

The double transitivity means that G acts transitively on the configuration space $\mathcal{C}_n^{[2]}$ of (ordered) pairs of points in $\mathcal{C}_n$; in other words, the diagonal action of G on $\mathcal{C}_n \times \mathcal{C}_n$ has exactly two orbits: the diagonal $\Delta = \{(p,p) \in \mathcal{C}_n \times \mathcal{C}_n\}$ and its complement $\mathcal{C}_n^{[2]} = (\mathcal{C}_n \times \mathcal{C}_n) \setminus \Delta$. One important consequence of this is that the stabilizer of each point in $\mathcal{C}_n$ (in particular, G_n) is a maximal subgroup of G. Theorem 1 thus strengthens the main results of [W2] and [KT], where it is shown that G_n coincides with its normalizer in G. A notable consequence of Theorem 2 is that the restriction of the action of G to G_n is transitive on $\mathcal{C}_k$ provided $k \neq n$.

We actually expect that Theorem 1 and Theorem 2 are part of the much stronger

Conjecture. *For any pairwise distinct natural numbers $(n_1, n_2, \ldots, n_m) \in \mathbb{N}^m$ and for any $(k_1, k_2, \ldots, k_m) \in \mathbb{N}^m$, the group G acts transitively on*

$$\mathcal{C}_{n_1}^{[k_1]} \times \mathcal{C}_{n_2}^{[k_2]} \times \ldots \times \mathcal{C}_{n_m}^{[k_m]} .$$

Here $\mathcal{C}_n^{[k]}$ stands for the configuration space of ordered k points in $\mathcal{C}_n$. The above conjecture implies, in particular, that G acts *infinitely* transitively on each $\mathcal{C}_n$, which is a well-known fact for $n = 1$. To put this in proper perspective we recall that the varieties $\mathcal{C}_n$ are examples of quiver varieties in the sense of Nakajima [N1]. Using the formalism of noncommutative symplectic geometry, V. Ginzburg [G] showed that the main theorem of [BW] holds for an arbitrary affine quiver variety $\mathcal{C}_{\boldsymbol{n}}(Q)$ in a weaker form: there is an infinite-dimensional Lie algebra[2] $\mathfrak{L}(Q)$ (canonically attached to the quiver Q) which acts *infinitesimally* transitively on $\mathcal{C}_{\boldsymbol{n}}(Q)$; in fact, each $\mathcal{C}_{\boldsymbol{n}}(Q)$ embeds in the dual of $\mathfrak{L}(Q)$ as a coadjoint orbit. The results of the present paper suggest that Ginzburg's theorem may admit a natural extension to higher configuration spaces $\mathcal{C}_{\boldsymbol{n}}^{[k]}(Q)$ and their products. For the further discussion of the above conjecture and its implications we refer to Section 3.6 and Section 5.4.

In the second part of the paper we will study $\{G_n\}$ as ind-algebraic groups. Recall that the notion of an ind-algebraic group goes back to I. Shafarevich who called such objects simply infinite-dimensional groups (see [Sh1, Sh2]). The fundamental example is the group $\mathrm{Aut}(\mathbb{C}^d)$ of polynomial automorphisms of the affine d-space. This group (sometimes called the affine Cremona group) has been extensively studied, especially for $d = 2$ (see, e.g., [J, vdK, Da, GD, Wr, K1, K2, FuL, FuM]). It is known [Cz, ML1] that as a *discrete* group, $\mathrm{Aut}(\mathbb{C}^2)$ is actually isomorphic to the automorphism group of the free algebra $\mathbb{C}\langle x, y\rangle$ and hence contains each G_n

[2]In the Calogero-Moser case, the Lie algebra $\mathfrak{L}(Q)$ is isomorphic to a central extension of the Lie algebra $\mathrm{Der}_w(R)$ of symplectic derivations of R. In Section 5.4, following the original suggestion of [BW], we will show that $\mathrm{Der}_w(R)$ can be identified with the Lie algebra of the group G equipped with an appropriate affine ind-scheme structure.

as a discrete subgroup. However, the ind-algebraic structure that we put on G_n is *different* (i.e., not induced) from $\mathrm{Aut}(\mathbb{C}^2)$. This ind-algebraic structure was originally proposed by G. Wilson and the first author in [BW], but the details were not worked out in that paper. It is interesting to note that the ind-algebraic group G can be defined in a simpler and somewhat more natural way than $\mathrm{Aut}(\mathbb{C}^2)$ and $\mathrm{Aut}(A_1)$. The reason for this is the remarkable fact [Di] that (the analogue of) the Jacobian Conjecture is known to be true for $\mathbb{C}\langle x, y\rangle$, while it is still open for the polynomial ring $\mathbb{C}[x, y]$ and the Weyl algebra $A_1(\mathbb{C})$.

Solvable subgroups play a key role in the theory of classical linear algebraic groups (see [Bo1]) as well as Kac-Moody groups [Ku]. It is therefore natural to expect that they should also play a role in the theory of ind-algebraic groups. In this paper, we will study the Borel subgroups of G_n: our main result is a complete classification of such subgroups for all n. To begin with, we recall that a Borel subgroup of a topological group is a connected solvable subgroup that is maximal among all connected solvable subgroups. The group G has an obvious candidate: the subgroup B of triangular[3] transformations: $(x, y) \mapsto (ax + q(y), a^{-1}y + b)$, where $q(y) \in \mathbb{C}[y]$, $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. It is not difficult to prove that B is indeed a Borel subgroup of G; moreover, as in the finite-dimensional case, we have the following theorem.

Theorem 3. *Any Borel subgroup of G is conjugate to B.*

For $n > 0$, the situation is more interesting. Let $\mathfrak{B}_n$ denote the set of all Borel subgroups of G_n on which G_n acts by conjugation. We will show that every Borel subgroup of G_n is conjugate in G to a subgroup of B: this defines a G_n-equivariant map $\iota : \mathfrak{B}_n \to B\backslash G$, where G_n acts on $B\backslash G$ by right multiplication. It turns out that, at the quotient level, the map ι induces a canonical injection

$$\mathfrak{B}_n/\mathrm{Ad}\, G_n \hookrightarrow \mathcal{C}_n/B . \tag{1}$$

Thus, the Borel subgroups of G_n are classified (up to conjugation) by orbits in $\mathcal{C}_n$ of the Borel subgroup of G. In general (more precisely, for $n \geq 2$), the map (1) is not surjective — not every B-orbit in $\mathcal{C}_n$ corresponds to a Borel subgroup of G_n — however, the image of (1) has a nice geometric description in terms of the $\mathbb{C}^*$-action on $\mathcal{C}_n$. To be precise, let $T := \{(ax, a^{-1}y) : a \in \mathbb{C}^*\} \subset B$ denote the group of scaling automorphisms, which is a maximal torus in G. We will prove

Theorem 4. *A B-orbit $\mathcal{O}$ in $\mathcal{C}_n$ corresponds to a conjugacy class of Borel subgroups in G_n if and only if one of the following conditions holds:*

(A) *T acts freely on $\mathcal{O}$.*
(B) *T has a fixed point in $\mathcal{O}$.*

The orbits of type (A) correspond precisely to the abelian Borel subgroups of G_n, while the orbits of type (B) correspond to the non-abelian ones.

Each of the two possibilities of Theorem 4 actually occurs: the orbits of type (A) exist in $\mathcal{C}_n$ for $n \geq 3$, while the orbits of type (B) exist for all n. Thus, in general, G_n has both abelian and non-abelian Borel subgroups. While the existence of abelian Borel subgroups remains mysterious to us, we have a fairly good understanding of the non-abelian ones. It is known (see [W]) that the T-fixed points in $\mathcal{C}_n$ are represented by nilpotent matrices (X, Y) and the latter are classified by the

[3] also known as de Jonquières transformations in the commutative case

partitions of n. We will show that the T-fixed points actually belong to *distinct* B-orbits, which are closed in $\mathcal{C}_n$. Thus Theorem 4 implies

Theorem 5. *The conjugacy classes of non-abelian Borel subgroups of G_n are in bijection with the partitions of n. In particular, for each $n \geq 0$, there are exactly $p(n)$ conjugacy classes of non-abelian Borel subgroups in G_n.*

The last result that we want to state in the Introduction provides an abstract group-theoretic characterization of non-abelian Borel subgroups of G_n.

Theorem 6. *An non-abelian subgroup H of G_n is Borel if and only if*

(B1) *H is a maximal solvable subgroup of G.*
(B2) *H contains no proper subgroups of finite index.*

Theorem 6 is an infinite-dimensional generalization of a classical theorem of R. Steinberg [St] that characterizes (precisely by properties (B1) and (B2)) the Borel subgroups in reductive affine algebraic groups. However, unlike in the finite-dimensional case, Steinberg's characterization does not seem to extend to *all* Borel subgroups of G_n (in fact, even for $n = 0$, there exist abelian subgroups that satisfy (B1) and (B2) but are countable and hence totally disconnected in G).

Theorem 5 and Theorem 6 combined together imply the following important

Corollary 1. *The groups G_n are pairwise non-isomorphic* (*as abstract groups*).

In fact, the groups G_n are distinguished from each other by the sets of conjugacy classes of their non-abelian Borel subgroups: by Theorem 5, these sets are finite and distinct, while by Theorem 6, they are independent of the algebraic structure.

Although the Borel subgroups of G_n have geometric origin and their classification is given in geometric terms, our proofs of Theorem 4 and Theorem 6 are not entirely geometric nor algebraic. The crucial ingredient is Friedland-Milnor's classificaition of polynomial automorphisms of $\mathbb{C}^2$ according to their dynamical properties (see [FM]). This classification was refined by Lamy [L] who extended it to a classification of subgroups of $\mathrm{Aut}(\mathbb{C}^2)$. We will identify G as a *discrete* group with $\mathrm{Aut}(\mathbb{C}^2)$ and use Lamy's classification as a main tool to study the subgroups of G.

In the end, we mention that the original goal of the present paper was to prove the result of Corollary 1. Our interest in this result is motivated by the following generalization of the Dixmier Conjecture (for A_1) proposed in [BEE].

Conjecture. *For all $n, m \geq 0$,*

$$\mathrm{Hom}(D_n,\, D_m) = \begin{cases} \varnothing & \text{if} \quad n \neq m \\ G_n & \text{if} \quad n = m \end{cases} \tag{2}$$

where 'Hom' *is taken in the category of unital associative* $\mathbb{C}$*-algebras.*

Corollary 1 implies that the endomorphism monoids $\mathrm{Hom}(D_n, D_n)$ are pairwise non-isomorphic for different n. Still, we do not know whether the above conjecture is actually stronger than the original Dixmier Conjecture which is formally the special case of (2) corresponding to $n = m = 0$.

The paper is organized as follows. In Section 2, we introduce notation, review basic facts about the Calogero-Moser spaces, the Weyl algebra and automorphism groups. This section contains no new results (except, possibly, for the proof of Theorem 10, which has not appeared in the literature).

In Section 3, after recalling elementary facts about doubly transitive actions, we prove Theorem 1 (Section 3.3) and Theorem 2 (Section 3.4). The main consequences of these theorems are discussed in Section 3.5 and related conjectures in Section 3.6.

In Section 4, we describe the structure of G_n as a discrete group, using the Bass-Serre theory of groups acting on graphs. The main result of this section (Theorem 12) gives an explicit presentation of G_n in terms of generalized amalgamated products. This result can be viewed as a generalization of the classical theorem of Jung and van der Kulk on the amalgamated structure of G.

In Section 5, we study G_n as ind-algebraic groups. After a brief review of ind-varieties and ind-groups in Section 5.1, we define the structure of an ind-group on G in Section 5.2 and on G_n (for $n \geq 1$) in Section 5.3. We show that G is connected (Theorem 13) and acts algebraically on $\mathcal{C}_n$ (Theorem 15). The connectedness of G_n for $n > 0$ is a more subtle issue: we prove that G_n is connected for $n = 1$ and $n = 2$ (Proposition 7) but leave it as a conjecture in general. In Section 5.4, we define another natural ind-algebraic structure on G that makes it an affine ind-group scheme $\mathcal{G}$. We show that $\mathcal{G} \ncong G$ as ind-schemes (Proposition 8) and identify the Lie algebra of $\mathcal{G}$ in terms of derivations of R, confirming a suggestion of [BW].

The main results of the paper are proved in Section 6. Specifically, Theorem 3 and Theorem 6 (for $n = 0$) are proved in Section 6.3, where we study the Borel subgroups of G. Theorem 4 is proved in Section 6.4, while Theorems 5 and 6 (for $n \geq 1$) in Section 6.5. Finally, in Section 6.6, we give a geometric construction of Borel subgroups in terms of singular curves and Wilson's adelic Grassmannian (see Proposition 13 and Corollary 11). We also give a complete list of representatives of the conjugacy classes of non-abelian Borel subgroups of G_n for $n = 1, 2, 3, 4$.

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2. Preliminaries

In this section, we fix notation and review basic facts from the literature needed for the present paper.

2.1. **The Calogero-Moser spaces.** For an integer $n \geq 1$, let $\mathcal{M}_n(\mathbb{C})$ denote the space of complex $n \times n$ matrices. Let $\tilde{\mathcal{C}}_n \subseteq \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C})$ be the subvariety of pairs of matrices (X, Y) satisfying the equation

$$\operatorname{rank}([X,\, Y] + I_n) = 1 \ , \tag{3}$$

where I_n is the identity matrix in $\mathcal{M}_n(\mathbb{C})$. It is easy to see that $\tilde{\mathcal{C}}_n$ is stable under the diagonal action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C})$ by conjugation of matrices, and the induced action of $\mathrm{PGL}_n(\mathbb{C})$ on $\tilde{\mathcal{C}}_n$ is free. Following [W], we define the *n-th Calogero-Moser space* to be the quotient variety $\mathcal{C}_n := \tilde{\mathcal{C}}_n/\mathrm{PGL}_n(\mathbb{C})$. It is shown in [W] that $\mathcal{C}_n$ is a smooth irreducible affine variety of dimension $2n$.

It is convenient to make sense of $\mathcal{C}_n$ for $n = 0$: as in [W], we simply assume that $\mathcal{C}_0$ is a point, and with this convention, we set

$$\mathcal{C} := \bigsqcup_{n\geq 0} \mathcal{C}_n \ .$$

Abusing notation, we will write $(X,\,Y)$ for a pair of matrices in $\tilde{\mathcal{C}}_n$ as well as for the corresponding point (conjugacy class) in $\mathcal{C}_n$.

The Calogero-Moser spaces can be obtained by (complex) Hamiltonian reduction (*cf.* [KKS]): specifically,

$$\mathcal{C}_n \cong \mu^{-1}(I_n)/\mathrm{GL}_n(\mathbb{C}) \ , \tag{4}$$

where $\mu : T^*(\mathfrak{gl}_n \times \mathbb{C}^n) \to \mathfrak{gl}_n\,,\ (X, Y, v, w) \mapsto -[X, Y] + vw$ is the moment map corresponding to the symplectic action of GL_n on the cotangent bundle $T^*(\mathfrak{gl}_n \times \mathbb{C}^n)$. With natural identification $T^*(\mathfrak{gl}_n \times \mathbb{C}^n) \cong \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \times \mathbb{C}^n \times (\mathbb{C}^n)^*$, this action is given by

$$(X, Y, v, w) \mapsto (gXg^{-1},\, gYg^{-1},\, gv,\, wg^{-1})\,, \quad g \in \mathrm{GL}_n(\mathbb{C}) \ . \tag{5}$$

It is easy to see that the orbit of $(X,\,Y,\,v,\,w) \in \mu^{-1}(I_n)$ under (5) is uniquely determined by the conjugacy class of $(X,\,Y) \in \tilde{\mathcal{C}}_n$; whence the isomorphism (4).

The above construction shows that the Calogero-Moser spaces carry a natural symplectic structure. In fact, it is known that each $\mathcal{C}_n$ is a hyperkähler manifold, and the symplectic structure on $\mathcal{C}_n$ is just part of a hyperkähler structure (see [N, Sect. 3.2] and [W]). In this paper, we will not use the hyperkähler structure and will regard $\mathcal{C}_n$ simply as a complex variety.

2.2. **The group G and its action on $\mathcal{C}_n$.** Let $R = \mathbb{C}\langle x,\, y\rangle$ be the free associative algebra on two generators x and y. Denote by $\mathrm{Aut}(R)$ the automorphism group of R. Every $\sigma \in \mathrm{Aut}(R)$ is determined by its action on x and y: we will write σ as $(\sigma(x),\, \sigma(y))$, where $\sigma(x)$ and $\sigma(y)$ are noncommutative polynomials in R given by the images of x and y under σ. A fundamental theorem of Czerniakiewics [Cz] and Makar-Limanov [ML1] states that $\mathrm{Aut}(R)$ is generated by the affine automorphisms:

$$(ax + by + e,\, cx + dy + f) \ , \quad a,\, b,\, \ldots,\, f \in \mathbb{C} \ ,$$

and the triangular (Jonquière) automorphisms:

$$(ax + q(y),\, by + h) \ , \quad a,\, b \in \mathbb{C}^*,\ h \in \mathbb{C} \ , \quad q(y) \in \mathbb{C}[y] \ .$$

This fact is often stated by saying that every automorphism of R is tame.

In this paper, we will study a certain family $\{G_0,\, G_1,\, G_2,\, \ldots\}$ of subgroups of $\mathrm{Aut}(R)$ associated with Calogero-Moser spaces. The first member in this family, which we will often denote simply by G, is the group of *symplectic* automorphisms of R:

$$G = G_0 := \{\sigma \in \mathrm{Aut}(R) \,:\, \sigma([x, y]) = [x, y]\} \ . \tag{6}$$

The structure of this group is described by the following theorem which is a simple consequence of the Czerniakiewics-Makar-Limanov Theorem.

Theorem 7 ([Cz], [ML1])**.** *The group G is the amalgamated free product*

$$G = A *_U B \ , \tag{7}$$

where A is the subgroup of symplectic affine transformations:

$$(ax + by + e,\, cx + dy + f) \ , \quad a,\, b,\, \ldots,\, f \in \mathbb{C} \ , \quad ad - bc = 1 \ , \tag{8}$$

B is the subgroup of symplectic triangular transformations:

$$(ax + q(y),\, a^{-1}y + h)\ , \quad a \in \mathbb{C}^*,\ h \in \mathbb{C}\ , \quad q(y) \in \mathbb{C}[y]\ , \tag{9}$$

and U is the intersection of A and B in G:

$$(ax + by + e,\, a^{-1}y + h)\ , \quad a \in \mathbb{C}^*,\ b,\, e,\, h \in \mathbb{C}\ . \tag{10}$$

Theorem 7 can be deduced from the well-known result of Jung [J] and van der Kulk [vdK] on the structure of the automorphism group of the polynomial algebra $\mathbb{C}[x, y]$ in two variables. The key observation of [Cz] and [ML1] was the following

Proposition 1. *The natural projection $R \twoheadrightarrow \mathbb{C}[x, y]$ induces an isomorphism of groups $\mathrm{Aut}(R) \xrightarrow{\sim} \mathrm{Aut}\,\mathbb{C}[x, y]$. Under this isomorphism, G corresponds to the subgroup $\mathrm{Aut}_\omega\,\mathbb{C}[x, y]$ of Poisson automorphisms (i.e. those with Jacobian 1).*

Remark. Proposition 1 implies that the natural action of G on $\mathbb{C}^2$ is faithful; this allows one to identify G with a subgroup of $\mathrm{Aut}(\mathbb{C}^2)$ and view the elements of G as polynomial automorphisms of $\mathbb{C}^2$; we will use this identification in Section 6. For a detailed proof of the Jung-van der Kulk Theorem as well as Proposition 1 we refer to [Co] (see, *loc. cit.*, Theorem 6.8.6 and Theorem 6.9.3, respectively). A direct proof of Theorem 7 can be found in [Co1].

Theorem 7 implies that G is generated by the automorphisms

$$\Phi_p \,:=\, (x,\, y + p(x))\ , \qquad \Psi_q \,:=\, (x + q(y),\, y)\ , \tag{11}$$

where $p(x) \in \mathbb{C}[x]$ and $q(y) \in \mathbb{C}[y]$. We denote the corresponding subgroups of G by $G_x := \langle \Phi_p \,:\, p \in \mathbb{C}[x] \rangle$ and $G_y := \langle \Psi_q \,:\, q \in \mathbb{C}[y] \rangle$. These are precisely the stabilizers of x and y under the natural action of G on R.

Next, following [BW], we define an action of G on the Calogero-Moser spaces $\mathcal{C}_n$. First, thinking of pairs of matrices (X, Y) as points dual to the coordinate functions $(x, y) \in R$, we let G act on $\mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C})$ by

$$(X,\, Y) \mapsto (\sigma^{-1}(X),\, \sigma^{-1}(Y))\ , \quad \sigma \in G\ . \tag{12}$$

Since G preserves commutators, this action restricts to the subvariety $\tilde{\mathcal{C}}_n$ of $\mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C})$ defined by (3) and commutes with the conjugation-action by $\mathrm{PGL}_n(\mathbb{C})$. Hence (12) defines an action of G on $\mathcal{C}_n$. Note that, for $n = 1$, the action of G on $\mathcal{C}_1 = \mathbb{C}^2$ agrees with the natural one coming from Proposition 1.

Knowing the structure of the group G (more precisely, the fact that G is generated by the triangular automorphisms (11)), it is easy to see that G acts on $\mathcal{C}_n$ symplectically and algebraically. Much less obvious is the following fact.

Theorem 8 ([BW])**.** *For each $n \geq 0$, the action of G on $\mathcal{C}_n$ is transitive.*

Theorem 8 plays a crucial role in the present paper. First, we use this theorem to define the groups G_n for $n \geq 1$: we let G_n be the stabilizer of a point in $\mathcal{C}_n$ under the action of G. By transitivity, this determines G_n uniquely up to conjugation in G. To do computations it will be convenient for us to choose specific representatives in each conjugacy class $[G_n]$; to this end we fix a basepoint $(X_0,\, Y_0) \in \mathcal{C}_n$ with

$$X_0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad Y_0 = \begin{pmatrix} 0 & 1-n & 0 & \dots & 0 \\ 0 & 0 & 2-n & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{13}$$

and set

$$G_n := \mathrm{Stab}_G(X_0, Y_0)\ , \quad n \geq 1\ . \tag{14}$$

2.3. **The Calogero-Moser correspondence.** Next, we recall the connection between the Calogero-Moser spaces and the Weyl algebra $A_1(\mathbb{C}) := R/\langle xy - yx - 1\rangle$ described in [BW]. In his 1968 paper [D], Dixmier proved that the automorphism group of A_1 is generated by the same transformations (11) as the group G. This result was refined by Makar-Limanov [ML2] who showed that the analogue of Proposition 1 also holds for $A_1(\mathbb{C})$: namely, the natural projection $R \twoheadrightarrow A_1$ induces an isomorphism of groups

$$G \xrightarrow{\sim} \mathrm{Aut}(A_1)\ . \tag{15}$$

Identifying $G = \mathrm{Aut}(A_1)$ via (15), we will look at the action of G on the space of ideals of A_1. To be precise, let $\mathcal{R} = \mathcal{R}(A_1)$ denote the set of isomorphism classes of nonzero right ideals of A_1. The automorphism group of A_1 acts naturally on the set of all right ideals (one simply treats an ideal as a subspace of A_1), and this action is compatible with isomorphism. Thus, we get an action: $G \times \mathcal{R} \to \mathcal{R}\,, (\sigma, [M]) \mapsto [\sigma(M)]$. The following result is another main ingredient of the present paper.

Theorem 9 ([BW])**.** *There is a bijective map $\omega : \mathcal{C} \to \mathcal{R}$ which is equivariant under the action of G.*

Note that in combination with Theorem 9, Theorem 8 shows that $\omega(\mathcal{C}_n)$ are precisely the orbits of G in $\mathcal{R}$. The map ω can be described explicitly as follows (*cf.* [BC]). Recall that a point of $\mathcal{C}_n$ is represented by a pair of matrices (X, Y) satisfying the equation (3). Factoring $[X,\, Y] + I_n = vw$ with $v \in \mathbb{C}^n$ and $w \in (\mathbb{C}^n)^*$, we define the (fractional) right ideal

$$M(X,Y) = \det(X - x\, I_n)\, A_1 \,+\, \chi(X,Y) \cdot \det(Y - y\, I_n)\, A_1\ . \tag{16}$$

where $\chi(X,Y) := 1 + w\,(X - x\, I_n)^{-1}(Y - y\, I_n)^{-1}v$ is an element of the quotient field of A_1. Now, the assignment $(X,Y) \mapsto M(X,Y)$ induces a map from $\mathcal{C}_n$ to the set of isomorphism classes of ideals of A_1; amalgamating such maps for all n yields the required bijection $\omega : \mathcal{C} \xrightarrow{\sim} \mathcal{R}$. Substituting the matrices (13) in (16), we find that the basepoint $(X_0, Y_0) \in \mathcal{C}_n$ corresponds to (the class of) the ideal

$$M(X_0, Y_0) = x^n A_1 + (y + nx^{-1})\, A_1\ . \tag{17}$$

We will denote the ideal (17) by M_n and write $D_n := \mathrm{End}_{A_1}(M_n)$ for its endomorphism ring. Note that $D_0 = A_1$.

2.4. **Automorphism groups.** Theorem 9 allows one to translate algebraic questions about A_1 and its module category to geometric questions about the Calogero-Moser spaces and the action of G on these spaces. One important application of this theorem is a classification of algebras (domains) Morita equivalent to A_1. Briefly, by Morita theory, every such algebra can be identified with the endomorphism ring of a right ideal in A_1; by a theorem of Stafford (see [S]), two such endomorphism rings are isomorphic (as algebras) iff the classes of the corresponding ideals lie in the same orbit of $\mathrm{Aut}(A_1)$ in $\mathcal{R}$. Now, using Theorem 9, we can identify the orbits of $\mathrm{Aut}(A_1)$ in $\mathcal{R}$ with the Calogero-Moser spaces $\mathcal{C}_n$. Thus, the domains Morita equivalent to A_1 are classified (up to isomorphism) by the single integer $n \geq 0$: every such domain is isomorphic to the algebra D_n, and moreover $D_n \ncong D_m$ for $n \neq m$. This classification was originally established in [K] by a direct calculation; it has several interpretations and many interesting implications which the reader may find in [BW2]. We conclude this section by recording a proof of the following fact which is mentioned in passing in [BW2].

Theorem 10. *Let $[M] \in \mathcal{R}$ be the ideal class corresponding to a point $(X, Y) \in \mathcal{C}_n$ under the Calogero-Moser map ω. Then, there is a natural isomorphism of groups*

$$\mathrm{Stab}_G(X, Y) \xrightarrow{\sim} \mathrm{Aut}[\mathrm{End}_{A_1}(M)] \ .$$

In particular, for all $n \geq 0$,

$$G_n = \mathrm{Aut}(D_n) \ . \tag{18}$$

Proof. First, we note that $\mathrm{Aut}[\mathrm{End}_{A_1}(M)]$ can be naturally identified with a subgroup of $\mathrm{Aut}(A_1)$. To be precise, let $\mathrm{Pic}(A)$ denote the Picard group of a $\mathbb{C}$-algebra A. Recall that $\mathrm{Pic}(A)$ is the group of $\mathbb{C}$-linear Morita equivalences of the category of A-modules; its elements are represented by the isomorphism classes of invertible A-bimodules P. There is a natural group homomorphism $\alpha_A : \mathrm{Aut}(A) \to \mathrm{Pic}(A)$, taking $\tau \in \mathrm{Aut}(A)$ to the class of the bimodule $[{}_1A_\tau]$, and if D is a ring Morita equivalent to A, with a progenerator M, then there is a group isomorphism $\beta_M : \mathrm{Pic}(D) \xrightarrow{\sim} \mathrm{Pic}(A)$ given by $[P] \mapsto [M^* \otimes_D P \otimes_D M]$. Now, for $A := A_1$ and $D := \mathrm{End}_A(M)$, we have the following diagram

$$\begin{tikzcd} \mathrm{Aut}(D) \arrow[r, "\alpha_D"] \arrow[d, dotted, "i_M"'] & \mathrm{Pic}(D) \arrow[d, "\beta_M"] \\ \mathrm{Aut}(A) \arrow[r, "\alpha_A"] & \mathrm{Pic}(A) \end{tikzcd} \tag{19}$$

where β_M is an isomorphism and the two horizontal maps are injective. A theorem of Stafford (see [S], Theorem 4.7) implies that α_A is actually an isomorphism. Inverting this isomorphism, we define the embedding $i_M : \mathrm{Aut}(D) \hookrightarrow \mathrm{Aut}(A)$, which makes (19) a commutative diagram.

Now, writing $H := \mathrm{Stab}_G(X, Y)$, we have group homomorphisms

$$H \hookrightarrow G \xrightarrow{\sim} \mathrm{Aut}(A) \xleftarrow{i_M} \mathrm{Aut}(D) \ ,$$

where the first map is the canonical inclusion and the second is the Makar-Limanov isomorphism (15). We claim that the image of H in $\mathrm{Aut}(A)$ coincides with the image of i_M; this gives the required isomorphism $H \xrightarrow{\sim} \mathrm{Aut}(D)$. In view of Theorem 9, it suffices to show that

$$\mathrm{Im}(i_M) = \{\, \tau \in \mathrm{Aut}(A) \ : \ \tau(M) \cong M \,\} \ .$$

First, we prove the inclusion $\mathrm{Im}(i_M) \subseteq \{\tau \in \mathrm{Aut}(A) \,:\, \tau(M) \cong M\}$. Given $\sigma \in \mathrm{Aut}(D)$, $i_M(\sigma)$ is defined to be the (unique) automorphism $\tau \in \mathrm{Aut}(A)$ such that

$$ {}_1A_\tau \cong M^* \otimes_D ({}_1D_\sigma) \otimes_D M \quad \text{(as A-bimodules)} \tag{20}$$

The right-hand side of (20) can be identified with the subspace $M^*\sigma(M) \subseteq Q$ in the quotient field of A, and we denote by f the corresponding isomorphism

$$ {}_1A_\tau \overset{\sim}{\to} M^* \otimes_D ({}_1D_\sigma) \otimes_D M \overset{\sim}{\to} M^*\sigma(M) \ .$$

Then, for any $a \in M \subseteq A$, we have $f(a) = f(a.1) = a\, f(1)$. On the other hand, $f(a) = f(1.a) = f(1)\sigma(\tau^{-1}(a))$. Thus, writing $b = f(1)$, we see that

$$\tau(M) = b\sigma(M)b^{-1} = bMM^*\sigma(M)b^{-1} = bM \ .$$

Conversely, suppose that $\tau(M) = bM$ for some $b \in M^*$. Then $\tau(D) = \tau(MM^*) = bMM^*b^{-1} = bDb^{-1}$ in Q. If we let $\sigma := \mathrm{Ad}_b \circ \tau \in \mathrm{Aut}(D)$, where $\mathrm{Ad}_b :\, a \mapsto b^{-1}ab$, then it is easy to see that $\tau = i_M(\sigma)$. $\square$

Remark. The above proof shows that for $n = 0$ the isomorphism (18) specializes to (15). Theorem 10 can thus be viewed as an extension of the Dixmier-Makar-Limanov theorem about $\mathrm{Aut}(A_1)$ to algebras Morita equivalent to A_1.

3. Double Transitivity

We begin by recalling basic properties of doubly transitive group actions.

3.1. **Doubly transitive group actions.** Let X be a set of cardinality $|X| \geq 2$. An action of a group G on X is called *doubly transitive* if for any two pairs (x_1, x_2) and (y_1, y_2) of distinct elements in X, there is a $g \in G$ such that $g\,x_1 = y_1$ and $g\,x_2 = y_2$. In other words, G acts doubly transitively on X if the diagonal action of G on $X \times X$ is transitive outside the diagonal $\Delta \subset X \times X$.

Note that a doubly transitive group action is automatically transitive, but the converse is obviously not true. The next lemma provides some useful characterizations of doubly transitive actions.

Lemma 1. *Let G be a group acting on a set X with $|X| \geq 3$. Then the following are equivalent.*

1. *The action of G on X is doubly transitive.*
2. *For each $x \in X$, the stabilizer $\mathrm{Stab}_G(x)$ acts transitively on $X \setminus \{x\}$.*
3. *G acts transitively on X, and there exists $x_0 \in X$ such that $\mathrm{Stab}_G(x_0)$ acts transitively on $X \setminus \{x_0\}$.*
4. *G acts transitively on X, and $G = H \cup gHg^{-1}$, where H is the stabilizer of a point in X and $g \in G \setminus H$.*

Proof. We will prove only that (1) $\Leftrightarrow$ (2) and leave the rest as a (trivial) exercise to the reader. Fix $x \in X$ and choose any $y, z \in X$ such that x, y, z are pairwise distinct. (This is possible since $|X| \geq 3$.) Then, a doubly transitive action admits $g \in G$ moving $y \mapsto z$ while fixing x. This proves (1) $\Rightarrow$ (2). Conversely, assume that (2) holds. Consider two pairs (x_1, x_2) and (y_1, y_2) in $X \times X$ with $x_1 \neq x_2$ and $y_1 \neq y_2$. If $x_1 \neq y_2$, then we can use elements of $\mathrm{Stab}_G(x_1)$ and $\mathrm{Stab}_G(y_2)$ moving $(x_1, x_2) \mapsto (x_1, y_2) \mapsto (y_1, y_2)$. If $x_1 = y_2$, then we choose $z \neq x_1, y_1$ in X (again, such a z exists since $|X| \geq 3$) and use the elements of $\mathrm{Stab}_G(x_1)$, $\mathrm{Stab}_G(z)$ and $\mathrm{Stab}_G(y_1)$ to move $(x_1, x_2) \mapsto (x_1, z) \mapsto (y_1, z) \mapsto (y_1, y_2)$. $\square$

Corollary 2. *Suppose G acts doubly transitively on a set X. Then*
(a) *the stabilizer of any point of X is a maximal subgroup of G.*
(b) *any normal subgroup $N \lhd G$ acts on X either trivially or transitively.*

Proof. (a) Fix $x \in X$ and let $H = \mathrm{Stab}_G(x)$. If $H \subsetneqq K \leqq G$, then $H \cup HgH \subseteq K$ for any $g \in K \setminus H$. But Proposition 1(4) implies that $H \cup HgH = G$. Hence $K = G$.

(b) Suppose that N acts nontrivially on X: i. e., $h\,x \neq x$ for some $x \in X$ and $h \in N$. Pick any two distinct elements in X, say y and z. Then, by double transitivity, there is $g \in G$ such that $y = g\,x$ and $z = g\,(hx)$. It follows that $z = ghg^{-1}(g\,x) = ghg^{-1}y$ and $ghg^{-1} \in N$, so N acts transitively on X. □

Remark. The transitive group actions with maximal stabilizers are called primitive. The above Corollary shows that any doubly transitive action is primitive. The converse is not always true: for example, the natural action of the dihedral group D_n on the vertices of a regular n-gon is primitive for n prime but not doubly transitive if $n \geq 4$.

3.2. **Auxiliary results.** To prove Theorems 1 and 2 we will need a few technical results from the earlier literature. First, following [EG], we define the map

$$\Upsilon\,:\,\mathcal{C}_n \to \mathbb{C}^n/S_n \times \mathbb{C}^n/S_n\ , \quad (X, Y)\,\mapsto\,(\mathrm{Spec}(X),\,\mathrm{Spec}(Y))\ ,$$

assigning to the matrices $(X, Y) \in \mathcal{C}_n$ their eigenvalues. We can write $\Upsilon = (\Upsilon_1, \Upsilon_2)$, where Υ_1 and Υ_2 are the projections onto the first and second factors, respectively. The following fact is proved in [EG] (see *loc. cit.*, Prop. 4.15 and Theorem 11.16).

Theorem 11. *The map Υ is surjective.*

Next, we recall the subgroups G_x and G_y of G generated by the automorphisms $(x, y) \mapsto (x, y + p(x))$ and $(x, y) \mapsto (x + q(y), y)$ respectively, see (11). These are precisely the stabilizers of x and y under the natural action of G on $R = \mathbb{C}\langle x, y\rangle$. The following simple observation is essentially due to [BW] (see *loc. cit.*, Sect. 10).

Lemma 2. *Let $(X, Y) \in \mathcal{C}_n$.*
(1) *If X is diagonalizable, then G_x acts transitively on $\Upsilon_1^{-1}(\mathrm{Spec}\,X)$.*
(2) *If Y is diagonalizable, then G_y acts transitively on $\Upsilon_2^{-1}(\mathrm{Spec}\,Y)$.*

Proof. We will only prove (1) ; the proof of (2) is similar. Assume that X is diagonal with $\mathrm{Spec}(X) = \{\lambda_1, \ldots, \lambda_n\}$. Then, by [W, (1.14)], the eigenvalues λ_i are pairwise distinct, and $(X, Y) \in \mathcal{C}_n$ if and only if the matrix Y has a standard Calogero-Moser form with off-diagonal entries

$$Y_{ij}\ =\ (\lambda_i - \lambda_j)^{-1} \quad (i \neq j)\ .$$

Any two such matrices, say Y and Y', may differ only in their diagonal entries: let $(a_1, \ldots, a_n)$ and $(a'_1, \ldots, a'_n)$ be these diagonal entries. Then, by Lagrange's Interpolation, there is $p(x) \in \mathbb{C}[x]$ such that $p(\lambda_i) = a_i - a'_i$ for all i. The corresponding automorphism $(x,\,y + p(x)) \in G_x$ moves (X, Y) to (X, Y'). □

Now, for each $k \geq 0$, we introduce the following subgroups of G:

$$G_{k,x} \quad := \quad \{(x, y + x^k p(x)) \in G\ :\ p(x) \in \mathbb{C}[x]\,\}\ , \tag{21}$$

$$G_{k,y} \quad := \quad \{(x + y^k q(y), y) \in G\ :\ q(y) \in \mathbb{C}[y]\,\}\ . \tag{22}$$

Note that

$$G_x = G_{0,x} \supset G_{1,x} \supset G_{2,x} \supset \ldots \quad , \quad G_y = G_{0,y} \supset G_{1,y} \supset G_{2,y} \supset \ldots$$

and in general, for any $k \geq 0$, we have[4]

$$G_{k,x} = G_k \cap G_x \quad \text{and} \quad G_{k,y} = G_k \cap G_y \ . \tag{23}$$

Lemma 3. *Let $(X, Y) \in \mathcal{C}_n$, and let $k \geq 0$ be any integer.*

(1) *If $\det(X) \neq 0$, then $G_{k,x}(X, Y) = G_x(X, Y)$.*
(2) *If $\det(Y) \neq 0$, then $G_{k,y}(X, Y) = G_y(X, Y)$.*

Proof. We will only prove (1). Let $\chi = \chi(X)$ denote the characteristic polynomial of X. Since $\det(X) \neq 0$, we have $\gcd(x^k, \chi) = 1$ for all k. It follows that for each $k \geq 0$, there are f and g in $\mathbb{C}[x]$ such that $f\, x^k + g\, \chi = 1$. Hence, any $p \in \mathbb{C}[x]$ can be written in the form $p = p\, f\, x^k + p\, g\, \chi$. By the Cayley-Hamilton Theorem, evaluating p at X then yields $p(X) = p(X)\, f(X)\, X^k$, which shows that $G_x(X, Y) = G_{k,x}(X, Y)$ for any k. $\square$

In the rest of this section, we will use the following notation.

$$\begin{aligned}
\mathcal{C}_n^* &:= \{(X, Y) \in \mathcal{C}_n \mid \det(X) \neq 0 \text{ or } \det(Y) \neq 0\} \\
\mathcal{C}_{n,1}^{*,\mathrm{reg}} &:= \{(X, Y) \in \mathcal{C}_n \mid \det(X) \neq 0 \text{ and } X \text{ is diagonalizable}\} \\
\mathcal{C}_{n,2}^{*,\mathrm{reg}} &:= \{(X, Y) \in \mathcal{C}_n \mid \det(Y) \neq 0 \text{ and } Y \text{ is diagonalizable}\} \\
\mathcal{C}_n^{*,\mathrm{reg}} &:= \mathcal{C}_{n,1}^{*,\mathrm{reg}} \cup \mathcal{C}_{n,2}^{*,\mathrm{reg}}
\end{aligned}$$

With this notation, Lemma 2 and Lemma 3 combined together imply

Corollary 3. *Let $k \geq 0$ be any integer.*

(1) *If $(X, Y) \in \mathcal{C}_{n,1}^{*,\mathrm{reg}}$, then $G_{k,x}$ acts transitively on $\Upsilon_1^{-1}(\mathrm{Spec}\, X)$.*
(2) *If $(X, Y) \in \mathcal{C}_{n,2}^{*,\mathrm{reg}}$, then $G_{k,y}$ acts transitively on $\Upsilon_2^{-1}(\mathrm{Spec}\, Y)$.*

Combining Corollary 3 with Theorem 11, we get

Corollary 4. *Let $k \geq 0$ be any integer.*

(1) *If $(X, Y) \in \mathcal{C}_{n,1}^{*,\mathrm{reg}}$ then there is $\sigma \in G_{k,x}$ such that $\sigma(X, Y) = (X, Y_1)$, where Y_1 is a diagonalizable matrix with eigenvalues $(1, \ldots, n)$.*
(2) *If $(X, Y) \in \mathcal{C}_{n,2}^{*,\mathrm{reg}}$ then there is $\sigma \in G_{k,y}$ such that $\sigma(X, Y) = (X_1, Y)$, where X_1 is a diagonalizable matrix with eigenvalues $(1, \ldots, n)$.*

Proof. Indeed, by Theorem 11, the set $\Upsilon^{-1}(1, \ldots, n; \mathrm{Spec}\, X)$ is nonempty. Since it is a subset of $\Upsilon_1^{-1}(\mathrm{Spec}\, X)$, statement (1) follows from Corollary 3(1). Similarly, statement (2) is a consequence of Corollary 3(2). $\square$

The next lemma is a slight modification of an important result due to T. Shiota.

Lemma 4 (*cf.* [BW], Lemma 10.3)**.** *For any $(X, Y) \in \mathcal{C}_n$, there exist polynomials $q \in \mathbb{C}[x]$ and $r \in \mathbb{C}[y]$ such that $Y + q(X)$ and $X + r(Y)$ are nonsingular diagonalizable matrices: that is,*

$$G_y(X, Y) \cap \mathcal{C}_{n,1}^{*,\mathrm{reg}} \neq \varnothing \text{ and } G_x(X, Y) \cap \mathcal{C}_{n,2}^{*,\mathrm{reg}} \neq \varnothing \, .$$

Remark. Shiota's Lemma (as stated in [BW, Lemma 10.3]) claims the existence of a polynomial $r \in \mathbb{C}[y]$ such that $X + r(Y)$ a diagonalizable matrix. Adding an appropriate constant to such a polynomial ensures that $\det(X + r(Y) + c\, I_n) \neq 0$.

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[4] However, unlike G, the groups G_k are not generated by $G_{k,x}$ and $G_{k,y}$ if $k \geq 2$. See Section 4.3.3 below.

3.3. **Proof of Theorem 1.** In view of Theorem 8 and Lemma 1(3), it suffices to prove that G_n acts transitively on $\mathcal{C}_n \setminus \{(X_0, Y_0)\}$, where (X_0, Y_0) is the basepoint of $\mathcal{C}_n$ (see (13)). We will establish the following more general fact:

$$|G_k \backslash \mathcal{C}_n| = \begin{cases} 1\ , & \text{if } k \neq n \\ 2\ , & \text{if } \ k = n \end{cases} \tag{24}$$

In the proof of (24) we may (and will) assume that $k \geq n$ (indeed, we have $G_k \backslash \mathcal{C}_n = G_k \backslash G / G_n = \mathcal{C}_k / G_n$, and all the above statements hold true for the right action of G_n on $\mathcal{C}_k$). Note also that for $k = n = 1$, the claim (24) is obvious because $\mathcal{C}_1 = \mathbb{C}^2$ and G_1 contains $\mathrm{SL}_2(\mathbb{C})$ which acts on $\mathbb{C}^2$ linearly as in its natural (irreducible) representation.

We prove (24) in three steps. First, we show that $\mathcal{C}_n^*$ is part of a single orbit of G_k on $\mathcal{C}_n$ for any $k \geq 0$ (see Proposition 2 below). Second, we show that if $k > n$ then $G_k(X, Y) \ \cap \ \mathcal{C}_n^* \neq \varnothing$ for any $(X, Y) \in \mathcal{C}_n$ (see Proposition 3). Finally, for $k = n$, we show that $G_n(X, Y) \ \cap \ \mathcal{C}_n^* \neq \varnothing$ for any $(X, Y) \neq (X_0, Y_0)$ (see Proposition 4).

Proposition 2. *$\mathcal{C}_n^*$ is in a single orbit of G_k for any $k \geq 0$.*

Proof. By (23), $G_{k,x}$ and $G_{k,y}$ are subgroups of G_k for any k. We will prove that $\mathcal{C}_n^*$ lies in a single orbit of the group generated by these subgroups. To this end, we first show that $\mathcal{C}_n^{*,\mathrm{reg}}$ lies in a single orbit of $\langle G_{k,x}, G_{k,y} \rangle$ and then we prove that the $\langle G_{k,x}, G_{k,y} \rangle$-orbit of any $(X, Y) \in \mathcal{C}_n^*$ meets $\mathcal{C}_n^{*,\mathrm{reg}}$.

Let $(X_i, Y_i) \in \mathcal{C}_{n,2}^{*,\mathrm{reg}}$ for $i = 1, 2$. We will show that these two points can be connected by an element in $\langle G_{k,x}, G_{k,y} \rangle$. By Corollary 4, there are $\sigma_i \in G_{k,y}$ such that $\sigma_i(X_i, Y_i) \ = \ (\tilde{X}, \tilde{Y}_i)$, where $\tilde{X} \ = \ \mathtt{Diag}(1, \ldots, n)$ and $\tilde{Y}_i$ is the corresponding Calogero-Moser matrix similar to Y_i. Now, since $(\tilde{X}, \tilde{Y}_i) \in \mathcal{C}_{n,1}^{*,\mathrm{reg}}$, we may again apply Corollary 4 to get $\tau \in G_{k,x}$ such that $\tau(\tilde{X}, \tilde{Y}_1) \ = \ (\tilde{X}, \tilde{Y}_2)$. It follows that $\sigma_2^{-1}\, \tau\, \sigma_1(X_1, Y_1) \ = \ (X_2, Y_2)$. A similar argument works for any pair of points in $\mathcal{C}_{n,1}^{*,\mathrm{reg}}$.

Now, suppose $(X_1, Y_1) \in \mathcal{C}_{n,1}^{*,\mathrm{reg}}$ and $(X_2, Y_2) \in \mathcal{C}_{n,2}^{*,\mathrm{reg}}$. Again, using Corollary 4, we may find $\tau \in G_{k,x}$ such that $\tau(X_1, Y_1) \ = \ (X_1, \tilde{Y})$, where $\tilde{Y} = \mathtt{Diag}(1, \ldots, n)$. Since both $(X_1, \tilde{Y})$ and (X_2, Y_2) are now in $\mathcal{C}_{n,2}^{*,\mathrm{reg}}$, we get back to the previous case.

Finally, let $(X, Y) \in \mathcal{C}_n^*$. By Lemma 3, we know that $G_{k,y}(X, Y) = G_y(X, Y)$ for any k. On the other hand, by Lemma 4, the G_y-orbit of (X, Y) always meets $\mathcal{C}_{n,1}^{*,\mathrm{reg}}$. This completes the proof of Proposition 2. □

To move an arbitrary point of $\mathcal{C}_n$ to $\mathcal{C}_n^*$ we will have to use automorphisms of G_k which are composites of elements of $G_{k,x}$ and $G_{k,y}$. To this end, for $k \geq 2$ and $p \in \mathbb{C}[x]$, we define

$$\sigma_{k,p} := \ (x + y^{k-1}, y) \circ (x, y - p(x)) \circ (x - y^{k-1}, y)\ , \tag{25}$$

and

$$\tau_{k,p} \ := \ (x + y^{k-2} + y^{k-1}, y) \circ (x, y - p(x)) \circ (x - y^{k-2} - y^{k-1}, y)\ . \tag{26}$$

Now, let (X_0, Y_0) be the basepoint of $\mathcal{C}_k$ $(k \geq 2)$ represented by (13). Write $\alpha(x) \in \mathbb{C}[x]$ and $\beta(x) \in \mathbb{C}[x]$ for the minimal polynomials of the matrices $X_0 - Y_0^{k-1}$ and $X_0 - Y_0^{k-2} - Y_0^{k-1}$ respectively. A simple calculation shows

$$\alpha(x) \ = \ (-1)^k x^k - (k-1)! \tag{27}$$

$$\beta(x) \;=\; x^k + (-1)^k \frac{k!}{k-1} x + (-1)^{k-1}(k-1)! \tag{28}$$

Lemma 5. *For any $c(x) \in \mathbb{C}[x]$, we have $\sigma_{k,\,\alpha(x)c(x)} \in G_k$ and $\tau_{k,\,\beta(x)c(x)} \in G_k$.*

Proof. Write $X_1 := X_0 - Y_0^{k-1}$ and take any $p \in \mathbb{C}[x]$ such that $p(X_1) = 0$. Then $\sigma_{k,\,p}$ acts on (X_0, Y_0) by

$$(X_0, Y_0) \xrightarrow{(x+y^{k-1},y)} (X_1, Y_0) \xrightarrow{(x,y-p(x))} (X_1, Y_0) \xrightarrow{(x-y^{k-1},y)} (X_1{+}Y_0^{k-1}, Y_0) \;=\; (X_0, Y_0)\,.$$

This shows that $\sigma_{k,p} \in G_k$ for $p = \alpha(x)c(x)$. For $\tau_{k,\,p}$, the calculation is similar. □

We are now ready to prove our next proposition.

Proposition 3. *Let $(X, Y) \in \mathcal{C}_n$. If $k > n$, then $G_k\,(X, Y) \cap \mathcal{C}_n^* \neq \varnothing$.*

Proof. Let $\tilde{G}_k := \langle G_k, G_{k-1,y}\rangle$ be the subgroup of G generated by G_k and $G_{k-1,y}$. First, we prove

Claim 1: $G_k(X, Y) = \tilde{G}_k(X, Y)$.

It is clear that $G_k(X, Y) \subseteq \tilde{G}_k(X, Y)$, since $G_k \subseteq \tilde{G}_k$. To prove the opposite inclusion it suffices to show that $G_{k-1,y}(X_1, Y_1) \subseteq G_k(X, Y)$ for any $(X_1, Y_1) \in G_k(X, Y)$. This is equivalent to showing that, for any $d(y) \in \mathbb{C}[y]$, the automorphism $(x + d(y)y^{k-1}, y)$ maps (X_1, Y_1) to a point in $G_k(X, Y)$. Now, let $(X_1, Y_1) \in G_k(X, Y)$. Then, for the minimal polynomial $p_1(y) := \mu(Y_1)$, we have $\gcd(y^k, p_1(y)) = y^l$ for some $l \leq n < k$ (since $\deg(p_1) \leq n$). Hence we can find $a(y), b(y) \in \mathbb{C}[y]$ such that $a(y)\, y^k + b(y)\, p_1(y) = y^l$. Multiplying by $d(y)\ y^{k-l-1}$, we get

$$d(y)\, a(y)\, y^{2k-l-1} + d(y)\, b(y)\, p_1(y)\, y^{k-l-1} \;=\; d(y)\, y^{k-1}\ ,$$

which, in turn, implies

$$(x + d(y)y^{k-1},\, y)\, (X_1, Y_1) \;=\; (x + d(y)a(y)y^{2k-l-1},\, y)\, (X_1, Y_1)\,.$$

Now, since $2k - l - 1 \geq k$, we see that $(x + d(y)a(y)y^{2k-l-1}, y) \in G_k$. This finishes the proof of Claim 1.

Claim 2: $G_x \subset \tilde{G}_k$, or equivalently, $(x, y + d(x)) \in \tilde{G}_k$ for any $d(x) \in \mathbb{C}[x]$.

By Lemma 5, $\sigma_{k,\,p} \in G_k \subset \tilde{G}_k$ for $p = \alpha(x)\, c(x)$, where $\sigma_{k,\,p}$ is the automorphism of G defined in (25), $c(x) \in \mathbb{C}[x]$ is any polynomial and $\alpha(x)$ is given by (27). Since $(x \pm y^{k-1}, y) \in G_{k-1,y} \subset \tilde{G}_k$, we have $(x, y + \alpha(x)c(x)) \in \tilde{G}_k$. Now, as $\gcd(\alpha(x), x^k) = 1$, for any $d(x) \in \mathbb{C}[x]$, we can find $a(x), b(x) \in \mathbb{C}[x]$ such that $a(x)\,\alpha(x) + b(x)\, x^k \;=\; d(x)$. This shows that $(x, y + d(x))$ is the composition of the automorphisms $(x, y + a(x)\alpha(x))$ and $(x, y + b(x)x^k)$, both of which are in $\tilde{G}_k$. This proves Claim 2.

Now, combining Claim 1 and Claim 2, we see that $G_x(X, Y) \subseteq G_k(X, Y)$. On the other hand, $G_x(X, Y) \cap \mathcal{C}_n^* \neq \varnothing$ by Lemma 4. Proposition 3 follows. □

With Proposition 2 and Proposition 3, the proof of (24) for $k \neq n$ is complete. To prove (24) for $k = n$ we will look at the action of G_n in the complement of $\mathcal{C}_n^*$

in $\mathcal{C}_n$. Writing $\mu(X)$ for the minimal polynomial of a matrix X, we define

$$\begin{aligned}
\mathcal{C}_n^0 &:= \mathcal{C}_n \backslash \mathcal{C}_n^* = \{(X,Y) \in \mathcal{C}_n \mid \det(X) = \det(Y) = 0\} ,\\
\mathcal{C}_{n,1}^0 &:= \{(X,Y) \in \mathcal{C}_n^0 \mid \mu(X) = x^n \text{ and } \mu(Y) = y^n\} ,\\
\mathcal{C}_{n,2}^0 &:= \{(X,Y) \in \mathcal{C}_n^0 \mid \mu(X) \neq x^n \text{ and } \mu(Y) = y^n\} ,\\
\mathcal{C}_{n,3}^0 &:= \{(X,Y) \in \mathcal{C}_n^0 \mid \mu(Y) \neq y^n\} ,
\end{aligned}$$

so that

$$\mathcal{C}_n^0 = \mathcal{C}_{n,1}^0 \bigsqcup \mathcal{C}_{n,2}^0 \bigsqcup \mathcal{C}_{n,3}^0 .$$

Since $\mathcal{C}_1^0 = (0,0)$, in the proof of the next proposition we will assume $n \geq 2$.

Proposition 4. *Let* $(X,Y) \in \mathcal{C}_n^0 \setminus \{(X_0,Y_0)\}$. *Then* $G_n\,(X,Y) \cap \mathcal{C}_n^* \neq \varnothing$.

Proof. We will consider three cases corresponding to the above decomposition. In case I and case II, we will explicitly produce $\tau \in G_k$ such that $\tau(X,Y) \in \mathcal{C}_n^*$. Then, the last case will be proved by contradiction assuming the first two cases.

Case I. Let $(X,Y) \in \mathcal{C}_{n,1}^0$. By [W, Proposition 6.8], $\mathcal{C}_{n,1}^0$ consists of exactly n points $\{(X(n,i),Y_0) : i = 1,\dots,n\}$, with $i=1$ corresponding to the base point (X_0,Y_0). Our goal is to show that there exist $\phi_i \in G_n$ such that $\phi_i(X(n,i),Y_0) \in \mathcal{C}_n^*$ for $i = 2,\dots,n$.

Let $X_i' := X(n,i) - Y_0^{n-2} - Y_0^{n-1}$, and let $q_i(x)$ be the minimal polynomials of X_i' for $i = 1,\dots,n$. Note that $q_1(x) = \beta(x)$ for $k = n$ (see (28)), since $(X(n,1),Y_0) = (X_0,Y_0)$. It is easy to compute the polynomials $q_i(x)$ explicitly:

$$q_i(x) \;=\; \begin{cases} x^n + (-1)^n \frac{n!}{n-1} x + (-1)^{n+1}(n-1)! \,, & i = 1 \\ x^n + (-1)^{n-i}(i-1)!(n-i)! \,, & i = 2,\dots,n-1 \\ x^n + \frac{n!}{n-1} x + (n-1)! \,, & i = n \end{cases} \tag{29}$$

and verify that $\mathtt{gcd}(q_i(x),\beta(x)) = 1$ for all $i = 2,\dots,n$. Hence, there are polynomials $a_i(x)$, $b_i(x) \in \mathbb{C}[x]$ such that

$$a_i(x)q_i(x) + b_i(x)\beta(x) = 1 . \tag{30}$$

Furthermore, by Lemma 4, for $(X_i',Y_0) \in \mathcal{C}_n$, there are $s_i(x) \in \mathbb{C}[x]$ such that

$$\det(Y_0 - s_i(X_i')) \neq 0 \tag{31}$$

Using these polynomials, we set $c_i(x) := s_i(x)\,b_i(x)\,\beta(x)$ for each $i = 2,\dots,n$ and consider the corresponding automorphisms $\tau_{n,\,c_i(x)}$ defined as in (26). By Lemma 5, $\tau_{n,\,c_i(x)} \in G_n$ since $\beta(x)$ divides $c_i(x)$. On the other hand, by (30), $c_i(x) = s_i(x) - s_i(x)\,b_i(x)\,q_i(x)$, which implies $c_i(X_i') = s_i(X_i')$. Thus, if we apply $\tau_{n,\,c_i}$, which is an element of G_n, to the pair $(X(n,i),Y_0)$, we get

$$\tau_{n,c_i}(X(n,i),Y_0) \;=\; (X_i' + (Y_0 - s_i(X_i'))^{n-2} + (Y_0 - s_i(X_i'))^{n-1}, Y_0 - s_i(X_i')) .$$

Now, using (31), we conclude $\sigma_{n,c_i}(X(n,i),Y_0) \in \mathcal{C}_n^*$.

Case II. Let $(X,Y) \in \mathcal{C}_{n,2}^0$. Then, by [W, Proposition 6.11], we may assume that $Y = Y_0$ and

$$X \;=\; X(n,r) + \sum_{k=1}^{n-1} X_{(k)} ,$$

where $2 \le r \le n$ and $X_{(k)}$ is a matrix with only nonzero entries on the k-th diagonal. Let $\lambda \in \mathbb{C}^*$. Applying a transformation $Q_\lambda \in G_n$, which is a scaling transformation followed by conjugation by $\mathtt{Diag}(1, \lambda, \ldots, \lambda^{n-1})$, see [W, Eq. (6.5)], we get

$$Q_\lambda(X, Y_0) \;=\; (X_\lambda, Y_0) \;:=\; (X(n,r) + \sum_{k=1}^{n-1} \lambda^{k-1} X_{(k)},\; Y_0)\,.$$

If $\det(X_\lambda) \ne 0$ then $(X_\lambda, Y_0) \in \mathcal{C}_n^*$, and we are done. If $\det(X_\lambda) = 0$ then either $\mu(X_\lambda) = x^n$ or $\mu(X_\lambda) \ne x^n$. If $\mu(X_\lambda) = x^n$, then $(X_\lambda, Y_0) \in \mathcal{C}^0_{n,1}$, which brings us back to case I established above. If $\mu(X_\lambda) \ne x^n$, then $q_\lambda(x) := \mu(X_\lambda)$ satisfies $\gcd(q_\lambda(x), x^n) = x^l$ for some $l < n$, and we can find $a(x), b(x) \in \mathbb{C}[x]$ such that

$$b(x)\, x^{2n-l-1} = x^{n-1} - a(x)\, q_\lambda(x)\, x^{n-1-l}\,.$$

If we apply $(x, y + b(x)x^{2n-l-1}) \in G_{n,x}$ to (X_λ, Y_0), we get $(X_\lambda, Y_0 + X_\lambda^{n-1})$. Now,

$$\det(Y_0 + X_\lambda^{n-1}) = \det(Y_0 + X(n,r)^{n-1}) + \lambda\, O(\lambda) = (-1)^{r-1}(r-1)!(n-r)! + \lambda\, O(\lambda)\,.$$

Thus, if we choose λ small enough, this last determinant is nonzero, and $\tau := Q_\lambda \circ (x, y + b(x)x^{2n-l-1}) \in G_n$ moves $(X, Y) \in \mathcal{C}^0_{n,2}$ to $(X_\lambda, Y_0 + X_\lambda^{n-1}) \in \mathcal{C}^*_n$.

Case III. Let $(X, Y) \in \mathcal{C}^0_{n,3}$. It suffices to prove that $G_n(X, Y) \not\subseteq \mathcal{C}^0_{n,3}$. Assume the contrary. Then, for any $(X_1, Y_1) \in G_n(X, Y)$, we have $\gcd(\mu(Y_1), y^n) = y^l$, where $\mu(Y_1)$ is the minimal polynomial of Y_1 and $l < n$. Arguing as in Proposition 3, we can show that $G_n(X, Y) = \tilde{G}_n(X, Y)$, where $\tilde{G}_n := \langle G_n, G_{n-1,y} \rangle$, and hence $G_x(X, Y) \subseteq G_n(X, Y)$. Then, by Lemma 4, there is $r(x) \in \mathbb{C}[x]$ such that $(X, Y + r(X)) \in \mathcal{C}_n^{*,\mathrm{reg}}$. This contradicts our assumption that $G_n(X, Y) \subseteq \mathcal{C}^0_{n,3}$. □

3.4. **Proof of Theorem 2.** We will assume that $n_1 < n_2 < \ldots < n_m$ and argue inductively in m. For $m = 1$, Theorem 2 is precisely Theorem 8. For $m = 2$, Theorem 2 follows from (24): indeed, given any point $(P_1, P_2) \in \mathcal{C}_{n_1} \times \mathcal{C}_{n_2}$, we first use the transitivity of G on $\mathcal{C}_{n_1}$ to move (P_1, P_2) to $(P_0(n_1), P_2')$, where $P_0(n_1) = (X_0(n_1), Y_0(n_1))$ is the basepoint of $\mathcal{C}_{n_1}$, and then use the transitivity of $G_{n_1} = \mathrm{Stab}_G[P_0(n_1)]$ on $\mathcal{C}_{n_2}$ to move $(P_0(n_1), P_2')$ to $(P_0(n_1), P_0(n_2))$. Now, to extend this argument to any m we need the following proposition.

Let $I = \{k_1 < k_2 < \ldots < k_r\}$ be a collection of positive integers written in increasing order. Let $(X_0(k), Y_0(k)) \in \mathcal{C}_k$ be the basepoints (13) of the spaces $\mathcal{C}_k$ corresponding to $k \in I$, and let $G_I := \cap_{k \in I} G_k$ denote the intersection of the stabilizers of these basepoints in G.

Proposition 5. *If $k > n$ for all $k \in I$, then G_I acts transitively on $\mathcal{C}_n$.*

Before proving Proposition 5, we make one elementary observation.

Lemma 6. *There is a polynomial $p(y) = \sum_{j \in I} a_{j-1} y^{j-1} \in \mathbb{C}[y]$ such that*

$$\det[\tilde{X}_0(k)] \ne 0 \quad \textit{for all} \quad k \in I\,,$$

where $\tilde{X}_0(k) := X_0(k) + p(Y_0(k))$.

Proof. For a fixed $k \in I$ and a generic polynomial $p(y) = \sum_{j=1}^{k-1} a_j y^j$, we have

$$\det(\tilde{X}_0(k)) = (-1)^{k-1}(k-1)!\, a_{k-1} + f(a_1, a_2, \ldots, a_{k-2})\,,$$

where $f(a_1, a_2, \ldots, a_{k-2})$ is some polynomial in $a_1, \ldots, a_{k-2}$. Applying this to each $k \in I$ and taking into account the fact that $Y_0(k)$ are nilpotent matrices of order k, we get r polynomials in r variables of the form

$$\det(\tilde{X}_0(k_s)) = a_{k_s} + f_{s-1}(a_{k_1}, \ldots, a_{k_{s-1}}) \ , \quad s = 1, 2, \ldots, r \ .$$

Such polynomials define an invertible transformation of the affine space $\mathbb{C}^r$, so we can certainly choose $a_{k_1}, \ldots, a_{k_r} \in \mathbb{C}$ in such a way that all $\det(\tilde{X}_0(k_s))$ take non-zero values. □

As a consequence of Lemma 6, we get

Corollary 5. *Let $p(y)$ and $\tilde{X}_0(k)$ be as in Lemma 6, and let $\mu_k(x) \in \mathbb{C}[x]$ denote the minimal polynomial of $\tilde{X}_0(k)$.*

(1) *The polynomial $\mu(x) := \prod_{k\in I} \mu_k(x)$ has a nonzero constant term.*
(2) *The automorphism $(x - p(y), y) \circ (x, y - \mu(x)c(x)) \circ (x + p(y), y)$ is in G_I for any $c(x) \in \mathbb{C}[x]$.*

Now, we can proceed with

Proof of Proposition 5. Note that G_I contains two abelian subgroups $G_{N,x} := \cap_{k\in I} G_{k,x}$ and $G_{N,y} := \cap_{k\in I} G_{k,y}$, where $N = \max\{k_1, \ldots, k_r\}$. We will argue as in Proposition 3. Define $\tilde{G}_I := \langle G_I, G_{n,y}\rangle$. We will show $G_I(X,Y) = \tilde{G}_I(X,Y) = G(X,Y)$ for any $(X,Y) \in \mathcal{C}_n$, and the proposition will follow from Theorem 8.

Claim 1: $G_I(X,Y) = \tilde{G}_I(X,Y)$.

First, $G_I(X,Y) \subseteq \tilde{G}_I(X,Y)$, since G_I is a subgroup of $\tilde{G}_I$. To prove the opposite inclusion it suffices to show that for any $d(y) \in \mathbb{C}[y]$, the automorphism $(x + d(y)y^n, y)$ preserves $G_I(X,Y)$. Let $(X_1, Y_1) \in G_I(X,Y)$, and let $p_1(y) := \mu(Y_1)$ be the minimal polynomial of Y_1. Then, since $\deg(p_1) \le n$, we have $\gcd(y^N, p_1(y)) = y^l$ for some $l \le n < N$. Hence, we can find $a(y), b(y) \in \mathbb{C}[y]$ such that $a(y)y^N + b(y)p_1(y) = y^n$. It follows that for any $d(y) \in \mathbb{C}[y]$,

$$(x + d(y)y^n, y)\,(X_1, Y_1) \,=\, (x + d(y)a(y)y^N, y)\,(X_1, Y_1)\,.$$

Thus $(x + d(y)y^n, y)$ maps (X_1, Y_1) into $G_{N,y}\,(X_1, Y_1) \subseteq G_I\,(X,Y)$.

Claim 2: $\tilde{G}_I = G$.

It suffices to show that $G_x \subset \tilde{G}_I$, or equivalently, $(x, y + d(x)) \in \tilde{G}_I$ for any $d(x) \in \mathbb{C}[x]$. Indeed, if $G_x \subset \tilde{G}_I$, conjugating $G_{N,y}$ by $(x, y+c)$ with an appropriate c we can show that G_y is also contained in $\tilde{G}_I$ and hence $G \subseteq G_I$ (since G_x and G_y generate G). Now, let $p(y) \in \mathbb{C}[y]$ be a polynomial defined in Lemma 6. Then, for any $c(x) \in \mathbb{C}[x]$, the composition of automorphisms $(x - p(y), y) \circ (x, y + \mu(x)c(x)) \circ (x + p(y), y)$ is in $\tilde{G}_I$ (see Corollary 5(2)). Since $(x \pm p(y), y) \in G_{n,y} \subset \tilde{G}_I$, we have $(x, y + \mu(x)c(x)) \in \tilde{G}_I$ for any $c(x) \in \mathbb{C}[x]$. By Corollary 5(1), $\gcd(\mu(x), x^N) = 1$, so for any $d(x) \in \mathbb{C}[x]$, we can find $a(x), b(x) \in \mathbb{C}[x]$ such that $a(x)\,\mu(x) + b(x)\,x^N = d(x)$. This shows that $(x, y + d(x))$ is the composition of $(x, y + a(x)\mu(x))$ and $(x, y + b(x)x^N)$, both of which are in $\tilde{G}_I$. It follows that $G_x \subseteq \tilde{G}_I$. This completes the proof of Claim 2, Proposition 5 and Theorem 2. □

3.5. **Corollary.** Theorem 1 and Theorem 2 have interesting implications. Recall that $G_k \subseteq G$ denotes the stabilizer of a point of $\mathcal{C}_k$ under the action of G, see (14).

Corollary 6. *Let k, n be non-negative integers.*

(1) *If $k \neq n$, then G_k acts transitively on $\mathcal{C}_n$. In this case, $G = G_k G_n$.*
(2) *For each $k \geq 0$, G_k is a maximal subgroup of G.*
(3) *The normalizer of G_k in G is equal to G_k. Moreover, if $k \neq n$, there is no $g \in G$ such that $g^{-1}\, G_k\, g \subseteq G_n$.*

Proof. (1) is an easy consequence of Theorem 2. (2) follows from Theorem 1 and Corollary 2(a). (3) is immediate from (2). □

If we reformulate Corollary 6 in terms of automorphism groups of algebras Morita equivalent to $A_1(\mathbb{C})$ (see Theorem 10), part (3) answers a question of Stafford [S]. As mentioned in the Introduction, this result was established by different methods in [KT] and [W2]. By Theorem 8, it is equivalent to the fact that the identity map is the *only* G-equivariant map from $\mathcal{C}_k$ to $\mathcal{C}_k$, and there are no G-equivariant maps $\mathcal{C}_k \to \mathcal{C}_n$ for $k \neq n$. In this form, Corollary 6(3) was proven in [W2]. Theorem 1 can thus be viewed as a strengthening of the main theorem of [W2].

3.6. **Infinite transitivity.** We conclude this section with two conjectures related to Theorems 1 and 2. For a space X and an integer $k > 0$, we denote by $X^{[k]}$ the configuration space of ordered k points of X, i.e.

$$X^{[k]} := \{(x_1, x_2, \ldots, x_k) \in X^k \,:\, x_i \neq x_j\}\ .$$

An action of a group G on X is then called *k-transitive* if the induced action $G \times X^{[k]} \to X^{[k]}$ is transitive. (Clearly, for $k = 2$, this definition agrees with the one given in Section 3.1.) A G-action which is k-transitive for all k, is called *infinitely transitive*. Since the natural projection $X^{[k]} \twoheadrightarrow X^{[k-1]}$ is G-equivariant, the k-transitivity implies the $(k-1)$-transitivity: in particular, any k-transitive action is transitive.

Conjecture 1. *For each $n \geq 1$, the action of G on $\mathcal{C}_n$ is infinitely transitive.*

Remarks. 1. Conjecture 1 is true for $n = 1$: this follows from a well-known (and elementary) fact that $\mathrm{Aut}(\mathbb{C}^d)$ acts infinitely transitively on $\mathbb{C}^d$ for all $d \geq 1$.

2. The infinite transitivity is an infinite-dimensional phenomenon: a finite-dimensional Lie group or algebraic group cannot act infinitely transitively on a variety. Indeed, if (say) an algebraic group H acts k-transitively on a variety X, there is a dominant map $H \to X^k$, $g \mapsto (g.x_1, \ldots, g.x_k)$ defined by a k-tuple $(x_1, \ldots, x_k)$ of pairwise distinct points of X; whence $\dim(H) \geq k \cdot \dim(X)$ (*cf.* [Bo2]). In Section 5, we will equip G with the structure of an infinite-dimensional algebraic group, which is compatible with the action of G on $\mathcal{C}_n$. Here, we note that no algebraic subgroup of G of finite dimension may act k-transitively on $\mathcal{C}_n$ if $k > 2$. Indeed, by Theorem 14, any finite-dimensional algebraic subgroup of G is conjugate to either a subgroup of A or a subgroup of B. In the first case, assuming that H acts k-transitively on $\mathcal{C}_n$, we have $2nk \leq \dim(H) \leq 5$, hence the H-action on $\mathcal{C}_n$ cannot be even transitive if $n > 2$ and it is at most 2-transitive in any case. In the second case, if H is conjugate to a subgroup of B: $(x, y) \mapsto (a^{-1}x + q(y),\, ay + b)$, the action may be at most doubly transitive, since so is the action of affine transformations $y \mapsto ay + b$ on $\mathbb{C}^1$.

3. In connection with Conjecture 1 we mention an interesting recent paper [AFKKZ]. For an affine variety X, the authors of [AFKKZ] introduce and study the group $\mathrm{SAut}(X)$ of *special* automorphisms of X. By definition, $\mathrm{SAut}(X)$ is generated by all one-parameter unipotent subgroups of $\mathrm{Aut}(X)$ (i.e. by the images in $\mathrm{Aut}(X)$ of the additive group $\mathbb{G}_a = (\mathbb{C}, +)$ coming from the regular actions $\mathbb{G}_a \times X \to X$). The main theorem (*cf.* [AFKKZ, Theorem 0.1]) implies that if X is smooth, the natural action of $\mathrm{SAut}(X)$ on X is transitive if and only if it is infinitely transitive. In view of Theorem 8, this result applies to our Calogero-Moser spaces $\mathcal{C}_n$, since the image of G in $\mathrm{Aut}(\mathcal{C}_n)$ under the action (12) lies in $\mathrm{SAut}(\mathcal{C}_n)$. To prove Conjecture 1 it would therefore suffice to show that G generates all of $\mathrm{SAut}(\mathcal{C}_n)$, which is certainly true for $n = 1$ (since $\mathcal{C}_1 \cong \mathbb{C}^2$) but seems unlikely for $n \geq 2$. Thus, Conjecture 1 may be viewed as a strengthening of the general results of [AFKKZ] in the special case $X = \mathcal{C}_n$.

The notion of infinite transitivity can be generalized in the following way (*cf.* [AFKKZ], Sect. 3.1). Let $X = \bigsqcup_n X_n$ be a disjoint union of G-sets (e.g., the orbits of an action of G on a space X). Then, for each integer $k > 0$, we can stratify

$$X^{[k]} = \bigsqcup_{k_1+\ldots+k_m=k} \ \bigsqcup_{n_1<\ldots<n_m} X_{n_1}^{[k_1]} \times \ldots \times X_{n_m}^{[k_m]} \ .$$

Now, we say that G acts *collectively infinitely transitively* on X if G acts transitively on each stratum $X_{n_1}^{[k_1]} \times \ldots \times X_{n_m}^{[k_m]}$ of $X_n^{[k]}$ for all $k > 0$. Intuitively, a collective infinite transitivity means the possibility to move simultaneously (i.e., by the same automorphism) an arbitrary finite collection of points from different orbits into a given position.

Theorem 1, Theorem 2 and Conjecture 1 are subsumed by the following general

Conjecture 2. *The action of G on $\mathcal{C} = \bigsqcup_{n\geq 0} \mathcal{C}_n$ is collectively infinitely transitive.*

Note that for $k = 1$, Conjecture 2 implies Theorem 8; for $k_1 = k$, it implies Conjecture 1, and for $m = k$ and $k_1 = k_2 = \ldots = k_m = 1$, it implies Theorem 2.

4. The Structure of G_n as a Discrete Group

In this section, we will use the Bass-Serre theory of graphs of groups to give an explicit presentation of G_n. We associate to each space $\mathcal{C}_n$ a graph Γ_n consisting of orbits of certain subgroups of G and identify G_n with the fundamental group $\pi_1(\mathbf{\Gamma}_n, *)$ of a graph of groups $\mathbf{\Gamma}_n$ defined by the stabilizers of those orbits in Γ_n. The Bass-Serre theory will then provide an explicit formula for $\pi_1(\mathbf{\Gamma}_n, *)$ in terms of generalized amalgamated products. The results of this section were announced in [BEE].

4.1. **Graphs of groups.** To state our results in precise terms we recall the notion of a graph of groups and its fundamental group (see [Se, Chapter I, §5]).

A *graph of groups* $\mathbf{\Gamma} = (\Gamma, G)$ consists of the following data:

(1) a connected graph Γ with vertex set $V = V(\Gamma)$, edge set $E = E(\Gamma)$ and incidence maps $i, t : E \to V$,

(2) a group G_a assigned to each vertex $a \in V$,

(3) a group G_e assigned to each edge $e \in E$,

(4) injective group homomorphisms $G_{i(e)} \stackrel{\alpha_e}{\hookleftarrow} G_e \stackrel{\beta_e}{\hookrightarrow} G_{t(e)}$ defined for each $e \in E$.

Associated to $\mathbf{\Gamma}$ is the path group $\pi(\mathbf{\Gamma})$, which is given by the presentation

$$\pi(\mathbf{\Gamma}) := \frac{(*_{a\in V} G_a) * \langle E\rangle}{(e^{-1}\alpha_e(g)\, e = \beta_e(g)\ :\ \forall\, e \in E,\, \forall\, g \in G_e)}\ ,$$

where '$*$' stands for the free product (i.e. coproduct in the category of groups) and $\langle E\rangle$ for the free group with basis set $E = E(\Gamma)$. Now, if we fix a maximal tree T in Γ, the *fundamental group* $\pi_1(\mathbf{\Gamma}, T)$ *of* $\mathbf{\Gamma}$ *relative to* T is defined as a quotient of $\pi(\mathbf{\Gamma})$ by 'contracting the edges of T to a point': precisely,

$$\pi_1(\mathbf{\Gamma}, T) := \pi(\mathbf{\Gamma})/(e = 1\ :\ \forall\, e \in E(T))\ . \tag{32}$$

For different maximal trees $T \subseteq \Gamma$, the groups $\pi_1(\mathbf{\Gamma},\, T)$ are isomorphic. Moreover, if $\mathbf{\Gamma}$ is trivial (i. e. $G_a = \{1\}$ for all $a \in V$), then $\pi_1(\mathbf{\Gamma},\, T)$ is isomorphic to the usual fundamental group $\pi_1(\Gamma,\, a_0)$ of the graph Γ viewed as a CW-complex. In general, $\pi_1(\mathbf{\Gamma},\, T)$ can be also defined in a topological fashion by introducing an appropriate notion of path and homotopy equivalence of paths in $\mathbf{\Gamma}$ (*cf.* [B], Sect. 1.6).

When the underlying graph of $\mathbf{\Gamma}$ is a tree (i.e., $\Gamma = T$), $\mathbf{\Gamma}$ can be viewed as a directed system of groups indexed by T. In this case, formula (32) shows that $\pi_1(\mathbf{\Gamma}, T)$ is just the inductive limit $\varinjlim \mathbf{\Gamma}$, which is called the *tree product* of groups $\{G_a\}$ amalgamated by $\{G_e\}$ along T. For example, if T is a segment with $V(T) = \{0,\, 1\}$ and $E(T) = \{e\}$, the tree product is the usual amalgamated free product $G *_{G_e} G_1$. In general, abusing notation, we will denote the tree product by

$$G_{a_1} *_{G_{e_1}} G_{a_2} *_{G_{e_2}} G_{a_3} *_{G_{e_3}} \ldots$$

4.2. G_n **as a fundamental group.** To define the graph Γ_n we take the subgroups A, B and U of G given by the transformations (8), (9) and (10), respectively. Restricting the action of G on $\mathcal{C}_n$ to these subgroups, we let Γ_n be the oriented bipartite graph, with vertex and edge sets

$$V(\Gamma_n) := (\mathcal{C}_n/A) \bigsqcup (\mathcal{C}_n/B)\ , \quad E(\Gamma_n) := \mathcal{C}_n/U\ , \tag{33}$$

and the incidence maps $E(\Gamma_n) \to V(\Gamma_n)$ given by the canonical projections $i : \mathcal{C}_n/U \to \mathcal{C}_n/A$ and $t : \mathcal{C}_n/U \to \mathcal{C}_n/B$. Since the elements of A and B generate G and G acts transitively on each $\mathcal{C}_n$, the graph Γ_n is connected.

Now, on each orbit in $\mathcal{C}_n/A$ and $\mathcal{C}_n/B$ we choose a basepoint and elements $\sigma_A \in G$ and $\sigma_B \in G$ moving these basepoints to the basepoint $(X_0,\, Y_0)$ of $\mathcal{C}_n$. Next, on each U-orbit $\mathcal{O}_U \in \mathcal{C}_n/U$ we also choose a basepoint and an element $\sigma_U \in G$ moving this basepoint to $(X_0,\, Y_0)$ and such that $\sigma_U \in \sigma_A A \cap \sigma_B B$, where σ_A and σ_B correspond to the (unique) A- and B-orbits containing $\mathcal{O}_U$. Then, we assign to the vertices and edges of Γ_n the stabilizers $A_\sigma = G_n \cap \sigma A \sigma^{-1}$, $B_\sigma = G_n \cap \sigma B \sigma^{-1}$, $U_\sigma = G_n \cap \sigma U \sigma^{-1}$ of the corresponding elements σ in the graph of right cosets of G under the action of G_n. These data together with natural group homomorphisms $\alpha_\sigma : U_\sigma \hookrightarrow A_\sigma$ and $\beta_\sigma : U_\sigma \hookrightarrow B_\sigma$ define a graph of groups $\mathbf{\Gamma}_n$ over Γ_n, and its fundamental group $\pi_1(\mathbf{\Gamma}_n,\, T)$ relative to a maximal tree $T \subseteq \Gamma_n$ has canonical presentation, cf. (32):

$$\pi_1(\mathbf{\Gamma}_n,\, T) = \frac{(A_\sigma *_{U_\sigma} B_\sigma * \ldots) * \langle\, E(\Gamma_n \setminus T)\,\rangle}{(\, e^{-1}\alpha_\sigma(g)\, e = \beta_\sigma(g)\ :\ \forall\, e \in E(\Gamma_n \setminus T),\, \forall\, g \in U_\sigma\,)}\ . \tag{34}$$

In (34), the amalgam $(A_\sigma *_{U_\sigma} B_\sigma * \ldots)$ stands for the tree product taken along the edges of T, while $\langle\, E(\Gamma_n\backslash\, T)\,\rangle$ denotes the free group generated by the set of edges of Γ_n in the complement of T. The main result of this section is the following

Theorem 12. *For each $n \geq 0$, the group G_n is isomorphic to $\pi_1(\mathbf{\Gamma}_n, T)$. In particular, G_n has an explicit presentation of the form (34).*

Proof. Let $\mathcal{G}_n := \mathcal{C}_n \rtimes G$ denote the (discrete) transformation groupoid corresponding to the action of G on $\mathcal{C}_n$. The canonical projection $p:\, \mathcal{G}_n \to G$ is then a covering of groupoids[5], which maps identically the vertex group of $\mathcal{G}_n$ at $(X_0,\, Y_0) \in \mathcal{C}_n$ to the subgroup $G_n \subseteq G$. Now, each of the subgroups A, B and U of G can be lifted to $\mathcal{G}_n$: $p^{-1}(A) = \mathcal{G}_n \times_G A$, $p^{-1}(B) = \mathcal{G}_n \times_G B$ and $p^{-1}(U) = \mathcal{G}_n \times_G U$, and these fibred products are naturally isomorphic to the subgroupoids $\mathcal{A}_n := \mathcal{C}_n \rtimes A$, $\mathcal{B}_n := \mathcal{C}_n \rtimes B$ and $\mathcal{U}_n := \mathcal{C}_n \rtimes U$ of $\mathcal{G}_n$, respectively. Since the coproducts of groups agree with coproducts in the category of groupoids and the latter can be lifted through coverings (see [O, Lemma 3.1.1]), the decomposition (7) implies

$$\mathcal{G}_n = \mathcal{A}_n *_{\mathcal{U}_n} \mathcal{B}_n\ , \quad \forall\, n \geq 0\ . \tag{35}$$

Unlike $\mathcal{G}_n$, the groupoids $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{U}_n$ are not transitive (if $n \geq 1$), so (35) can be viewed as an analogue of the Seifert-Van Kampen Theorem for nonconnected spaces. As in topological situation, computing the fundamental (vertex) group from (35) amounts to contracting the connected components (orbits) of $\mathcal{A}_n$ and $\mathcal{B}_n$ to points (vertices) and $\mathcal{U}_n$ to edges (see, e.g., [Ge, Chap. 6, Appendix]). This defines a graph which is exactly Γ_n. Now, choosing basepoints in each of the contracted components and assigning the fundamental groups at these basepoints to the corresponding vertices and edges defines a graph of groups (*cf.* [HMM], p. 46). By [HMM, Theorem 3], this graph of groups is (conjugate) isomorphic to the graph $\mathbf{\Gamma}_n$ described above, and our group G_n is isomorphic to $\pi_1(\mathbf{\Gamma}_n, T)$. $\square$

4.3. **Examples.** We now look at the graphs Γ_n and groups G_n for small n.

4.3.1. For $n = 0$, the space $\mathcal{C}_n$ is just a point, and so are a fortiori its orbit spaces. The graph Γ_0 is thus a segment, and the corresponding graph of groups $\mathbf{\Gamma}_0$ is given by $[\,A \xrightarrow{U} B\,]$. Formula (34) then says that $G_0 = A *_U B$ which agrees with (7).

4.3.2. For $n = 1$, we have $\mathcal{C}_1 \cong \mathbb{C}^2$, with $(X_0, Y_0) = (0,0)$. Since each of the groups A, B and U contains translations $(x+a, y+b)$, $a, b \in \mathbb{C}$, they act transitively on $\mathcal{C}_1$. So again Γ_1 is just the segment, and $\mathbf{\Gamma}_1$ is given by $[\,A_1 \xrightarrow{U_1} B_1\,]$, where $A_1 := G_1 \cap A$, $B_1 := G_1 \cap B$ and $U_1 := G_1 \cap U$. Since, by definition, G_1 consists of all $\sigma \in G$ fixing origin, the groups A_1, B_1 and U_1 are obvious:

$$\begin{aligned}
A_1:&\quad (ax+by,\, cx+dy)\ , \quad a,\, b,\, c,\, d \in \mathbb{C}\ ,\ ad-bc = 1\ ,\\
B_1:&\quad (ax+q(y),\, a^{-1}y)\ , \quad a \in \mathbb{C}^*\ ,\ q \in \mathbb{C}[y]\ ,\ q(0) = 0\ ,\\
U_1:&\quad (ax+by,\, a^{-1}y)\ , \quad a \in \mathbb{C}^*\ ,\ b \in \mathbb{C}\ .
\end{aligned}$$

It follows from (34) that $G_1 = A_1 *_{U_1} B_1$. In particular, G_1 is generated by its subgroups $G_{1,x}$ and $G_{1,y}$.

[5]Recall that if $\mathcal{E}$ and $\mathcal{B}$ are (small connected) groupoids, a covering $p:\mathcal{E}\to\mathcal{B}$ is a functor that is surjective on objects and restricts to a bijection $p: x\backslash\mathcal{E} \xrightarrow{\sim} p(x)\backslash\mathcal{B}$ for all $x \in \mathrm{Ob}(\mathcal{E})$, where $x\backslash\mathcal{E}$ is the set of arrows in $\mathcal{E}$ with source at x (see [M], Chap. 3).

4.3.3. The group G_2 has a more interesting structure. To describe the corresponding graph Γ_2 we decompose

$$\mathcal{C}_2 = \mathcal{C}_2^{\mathrm{reg}} \bigsqcup \mathcal{C}_2^{\mathrm{sing}} \ , \tag{36}$$

where $\mathcal{C}_2^{\mathrm{reg}}$ is the subspace of $\mathcal{C}_2$ with Y diagonalizable. The following lemma is established by elementary calculations.

Lemma 7. *The action of U on $\mathcal{C}_2$ preserves the decomposition* (36)*. Moreover,*
(a) $\mathcal{C}_2^{\mathrm{reg}}$ is a single U-orbit $\mathcal{O}^{\mathrm{reg}}$ passing through $(X_1, Y_1) \in \mathcal{C}_2$ with

$$X_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ .$$

(b) $\mathcal{C}_2^{\mathrm{sing}}$ consists of two orbits $\mathcal{O}(2,1)$ and $\mathcal{O}(2,2)$ passing through $(X(2,1), Y_2)$ and $(X(2,2), Y_2)$ with

$$X(2,1) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad , \quad X(2,2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad Y_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ .$$

Note that the orbit $\mathcal{O}^{\mathrm{reg}}$ is open (and therefore has dimension 4); $\mathcal{O}(2,1)$ and $\mathcal{O}(2,2)$ are closed orbits of dimension 3.

Lemma 8. *The B-orbits in $\mathcal{C}_2$ coincide with the U-orbits.*

Proof. Note that B belongs to the right coset $U\,\Psi_q$ generated by $\Psi_q = (x + q(y), y)$ with $q = ay^2 + by^3 + \ldots$. Since Y_2 is nilpotent, such Ψ_q acts trivially on $(X(2,r), Y_2)$, so $B\,(X(2,r), Y_2) = U\,(X(2,r), Y_2) = \mathcal{O}(2,r)$ for $r = 1, 2$. It follows that $\mathcal{O}(2,1)$ and $\mathcal{O}(2,2)$ are distinct B-orbits. Since there are only three U-orbits in $\mathcal{C}_2$, $\mathcal{O}^{\mathrm{reg}}$ must be a separate B-orbit. $\square$

Lemma 9. *The group A acts transitively on $\mathcal{C}_2$.*

Proof. Assume that A has more than one orbit in $\mathcal{C}_2$. Since there are only three U-orbits, at least one of the A-orbits (say, $\mathcal{O}_A$) consists of a single U-orbit. But then, by Lemma 8, $\mathcal{O}_A$ is also a B-orbit. Since A and B generate G, this means that $\mathcal{O}_A$ is a G-orbit and hence, by Theorem 9, coincides with $\mathcal{C}_2$. Contradiction. $\square$

Summing up, we have $\mathcal{C}_2/A = \{\mathcal{O}_A\}$ and

$$\mathcal{C}_2/B = \{\mathcal{O}_B^{\mathrm{reg}},\, \mathcal{O}_B(2,1),\, \mathcal{O}_B(2,2)\} \ , \quad \mathcal{C}_2/U = \{\mathcal{O}_U^{\mathrm{reg}},\, \mathcal{O}_U(2,1),\, \mathcal{O}_U(2,2)\} \ ,$$

where $\mathcal{O}_B$ and $\mathcal{O}_U$ denote the same subspaces in $\mathcal{C}_2$ but viewed as B- and U-orbits respectively. Thus the graph Γ_2 is a tree which looks as

$$\begin{tikzcd}
 & & \mathcal{O}_B(2,1) \\
\Gamma_2 \ : & \mathcal{O}_A \arrow[ur, "\mathcal{O}_U(2,1)"] \arrow[r, "\mathcal{O}_U^{\mathrm{reg}}"] \arrow[dr, "\mathcal{O}_U(2,1)"'] & \mathcal{O}_B^{\mathrm{reg}} \\
 & & \mathcal{O}_B(2,2)
\end{tikzcd}$$

Computing the stabilizers of basepoints for each of the orbits, we obtain the graph of groups

$$\mathbf{\Gamma}_2 \ : \qquad \begin{tikzcd} & G_{2,y} \rtimes T \\ T \arrow[ur, "T"] \arrow[r, "\mathbb{Z}_2"] \arrow[dr, "T"'] & G^{(1)}_{2,y} \rtimes \mathbb{Z}_2 \\ & G_{2,x} \rtimes T \end{tikzcd}$$

where $T \subset G$ is the subgroup of scaling transformations $(\lambda x, \lambda^{-1} y)$, $\lambda \in \mathbb{C}^*$, and the group $G^{(1)}_{2,y}$ is defined in terms of generators (11) by

$$G^{(1)}_{2,y} \quad := \quad \{\, \Phi_{-x}\, \Psi_q\, \Phi_x \in G \ : \ q \in \mathbb{C}[y]\ ,\ q(\pm 1) = 0 \,\}\ .$$

Formula (34) yields the presentation

$$G_2 = (G_{2,x} \rtimes T) *_T (G_{2,y} \rtimes T) *_{\mathbb{Z}_2} (G^{(1)}_{2,y} \rtimes \mathbb{Z}_2)\ . \tag{37}$$

In particular, G_2 is generated by its subgroups $G_{2,x},\ G_{2,y},\ G^{(1)}_{2,y}$ and T.

Using the above explicit presentations, it is easy to show that the groups G, G_1 and G_2 are pairwise non-isomorphic (see [BEE]). In Section 6, we will give a general proof of this fact for all groups G_n. For $n \geq 3$, the amalgamated structure of G_n seems to be more complicated; the corresponding graphs Γ_n are no longer trees (in fact, there are infinitely many cycles).

5. G_n as an algebraic group

In this section, we will equip G with the structure of an ind-algebraic group that is compatible with the action of G on the varieties $\mathcal{C}_n$. Each $G_n \subseteq G$ will then become a closed subgroup and hence will acquire an ind-algebraic structure as well. We begin by recalling the definition and basic properties of ind-algebraic varieties. Apart from the original papers of Shafarevich [Sh1, Sh2] a good reference for this material is Chapter IV of [Ku].

5.1. **Ind-algebraic varieties and groups.** An *ind-algebraic variety* (for short: an *ind-variety*) is a set $X = \bigcup_{k\geq 0} X^{(k)}$ given together with an increasing filtration

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots$$

such that each $X^{(k)}$ has the structure of a finite-dimensional (quasi-projective) variety over $\mathbb{C}$, and each inclusion $X^{(k)} \hookrightarrow X^{(k+1)}$ is a closed embedding of varieties. An ind-variety has a natural topology where a subset $S \subseteq X$ is open (resp, closed) iff $S^{(k)} := S \cap X^{(k)}$ is open (resp, closed) in the Zariski topology of $X^{(k)}$ for all k. In this topology, a closed subset S acquires an ind-variety structure defined by putting on $S^{(k)}$ the closed (reduced) subvariety structure from $X^{(k)}$. We call S equipped with this structure a *closed ind-subvariety* of X. More generally, any locally closed subset $S \subseteq X$ acquires from X the structure of an ind-variety since each $S^{(k)}$ is a locally closed subset[6] and hence a subvariety in $X^{(k)}$.

An ind-variety X is said to be *affine* if each $X^{(k)}$ is affine. For an affine ind-variety X, we define its *coordinate ring* by $\mathbb{C}[X] := \varprojlim_k \mathbb{C}[X^{(k)}]$, where $\mathbb{C}[X^{(k)}]$ is the coordinate ring of $X^{(k)}$. Naturally, $\mathbb{C}[X]$ is a topological algebra equipped with the inverse limit topology.

If X and Y are two ind-varieties with filtrations $\{X^{(k)}\}$ and $\{Y^{(k)}\}$, a map $f : X \to Y$ defines a *morphism* of ind-varieties if for each $k \geq 0$, there is $m \geq 0$ (depending on k) such that $f(X^{(k)}) \subseteq Y^{(m)}$ and the restriction of f to $X^{(k)}$, $f_k : X^{(k)} \to Y^{(m)}$, is a morphism of varieties. A morphism of ind-varieties $f : X \to Y$ is continuous with respect to ind-topology and, in the case of affine ind-varieties, induces a continuous algebra map $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$.

Example 1. For any ind-varieties X and Y, the set $X \times Y$ has the canonical ind-variety structure defined by the filtration $(X \times Y)^{(k)} := X^{(k)} \times Y^{(k)}$, where $X^{(k)} \times Y^{(k)}$ is a product in the category of varieties (*cf.* [Ku, Example 4.1.3 (2)]). The two natural projections $X \twoheadleftarrow X \times Y \twoheadrightarrow Y$ are then morphisms of ind-varieties.

A morphism of ind-varieties $f : X \to Y$ is called an *isomorphism* if f is bijective and f^{-1} is also a morphism. It is easy to see that a morphism $f : X \to Y$ of affine ind-varieties is an isomorphism iff the induced map $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ is an isomorphism of topological algebras. Two ind-variety structures on the same set X are said to be *equivalent* if the identity map $\mathrm{Id} : X \to X$ is an isomorphism. It is natural not to distinguish between equivalent structures on X.

Example 2. Any vector space V of countable dimension can be given the structure of an (affine) ind-variety by choosing a filtration $V^{(k)}$ by finite-dimensional subspaces. It is easy to see that up to equivalence, this structure is independent of the choice of filtration; hence V has the canonical structure of an ind-variety which is denoted $\mathbb{C}^\infty$ (*cf.* [Ku, Example 4.1.3 (4)]).

A morphism of ind-varieties $f : X \to Y$ is called a *closed embedding* if all the morphisms $f_k : X^{(k)} \to Y^{(m)}$ are closed embeddings, $f(X)$ is closed in Y and $f : X \to f(X)$ is a homeomorphism under the subspace topology on $f(X)$. The next lemma gives a useful characterization of morphisms of ind-varieties in terms of closed embeddings (*cf.* [Ku, Lemma 4.1.2]).

Lemma 10. *Let X, Y, Z be ind-varieties. Let $f : X \to Y$ be a closed embedding, and let $g : Z \to X$ be a map of sets with the property that for every $k \geq 0$ there is $m \geq 0$ such that $g(Z^{(k)}) \subseteq X^{(m)}$. Then f is a morphism (resp., closed embedding) iff $f \circ g : Z \to Y$ is a morphism (resp., closed embedding).*

For example, if $Z \subseteq Y$ is a closed ind-subvariety of Y, then the inclusion $Z \hookrightarrow Y$ is a closed embedding. Lemma 10 shows that the converse is actually also true:

Corollary 7. *If $Z \subseteq Y$ is a closed subset of an ind-variety Y, there is a unique ind-variety structure on Z making $Z \hookrightarrow Y$ a closed embedding.*

Proof. Assume that Z has two ind-variety structures, say Z' and Z'', making $Z \hookrightarrow Y$ into closed embeddings: $i' : Z' \to Y$ and $i'' : Z'' \to Y$. To apply Lemma 10

[6] The converse is not true: a subset $S \subset X$ may not be locally closed in X even though each of its components $S^{(k)}$ is locally closed in $X^{(k)}$. A counterexample is given in [FuM, Sect. 2.3].

we first take $f := i''$ and $g : Z' \to Z''$ to be the identity map Id_Z. Since $f \circ g = i'$ and $g = \mathrm{Id}$ obviously satisfies the assumption of the lemma, we conclude that $\mathrm{Id} : Z' \to Z''$ is a morphism of ind-varieties. Reversing the roles of Z' and Z'', we similarly conclude that $\mathrm{Id} : Z'' \to Z'$ is a morphism. Thus $Z' \cong Z''$. □

An *ind-algebraic group* (for short: an *ind-group*) is a group H equipped with the structure of an ind-variety such that the map $H \times H \to H,\ (x, y) \mapsto xy^{-1}$, is a morphism of ind-varieties. A *morphism of ind-groups* is an abstract group homomorphism which is also a morphism of ind-varieties. For example, any closed subgroup K of H is again an ind-group under the closed ind-subvariety structure on H, and the natural inclusion $K \hookrightarrow H$ is a morphism of ind-groups. Finally, an action of an ind-group H on an ind-variety H is said to be *algebraic* if the action map $H \times X \to X$ is a morphism of ind-varieties.

5.2. **The ind-algebraic structure on G.** Recall that R is the free associative algebra on two generators x and y. Letting V be the vector space spanned by x and y we identify R with the tensor algebra $T_{\mathbb{C}}(V) := \bigoplus_{n\geq 0} V^{\otimes n}$. Then, associated to the natural tensor algebra grading is a filtration on R by vector subspaces:

$$R^{(0)} \subseteq R^{(1)} \subseteq \ldots \subseteq R^{(k)} \subseteq R^{(k+1)} \subseteq \ldots\ , \tag{38}$$

where $R^{(k)} := \bigoplus_{n\leq k} V^{\otimes n}$. Since each $R^{(k)}$ has finite dimension, this filtration makes R an affine ind-variety, which is isomorphic to $\mathbb{C}^\infty$ (see Example 2). We write $\deg : R \to \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ for the degree function associated with (38): explicitly, if $p \in R$ is nonzero, $\deg(p) := k \Leftrightarrow p \in R^{(k)} \setminus R^{(k-1)}$, while $\deg(0) := -\infty$ by convention. Thus $R^{(k)} = \{p \in R : \deg(p) \leq k\}$.

Now, let $E := \mathrm{End}(R)$ denote the set of all algebra endomorphisms of R. Each endomorphism is determined by its values on x and y; hence we can identify

$$E = R \times R\ , \quad \sigma \mapsto (\sigma(x), \sigma(y))\ . \tag{39}$$

This identification allows us to equip E with an ind-variety structure by taking the product on ind-variety structures on R (see Example 1):

$$E^{(k)} := R^{(k)} \times R^{(k)} = \{(p, q) \in R \times R : \deg(p) \leq k\, , \deg(q) \leq k\}$$

Clearly, E is an affine ind-variety, which is actually isomorphic to $\mathbb{C}^\infty$. We define the degree function on E by $\deg(\sigma) := \max\{\deg(p), \deg(q)\}$, where $\sigma = (p, q) \in E$.

Next, recall that we have defined G to be the subset of E consisting of invertible endomorphisms that preserve $w = [x, y] \in R$. On the other hand, a well-known theorem of Dicks, which is an analogue of the Jacobian conjecture for R, implies that *every* endomorphism of R that preserve w is actually invertible (see [Co, Theorem 6.9.4]). We will use this result to put an ind-variety structure on G.

Proposition 6. *There is a unique ind-variety structure on G making the inclusion $G \hookrightarrow E$ a closed embedding.*

Proof. Consider the map

$$c : E \to R\ , \quad (p, q) \mapsto [p, q]\ , \tag{40}$$

where $[p, q] := p\,q - q\,p$ is the commutator in R. Since $c(E^{(k)}) \subseteq R^{(2k)}$ for all $k \geq 0$ and the restrictions $c_k : E^{(k)} \to R^{(2k)}$ are given by polynomial equations, (40) is a morphism of ind-varieties (in particular, a continuous map). Under the identification (39), we have $c(\sigma) = w$ for all elements $\sigma \in G$. Now, by Dicks'

Theorem, G actually coincides with the preimage of w. Since c is a continuous map (and $w \in R$ is a closed point), $G = c^{-1}(w)$ is a closed subset in E. Letting $G^{(k)} = G \cap E^{(k)}$ for $k \geq 0$ and putting on $G^{(k)}$ the closed (reduced) subvariety structure from $E^{(k)} = R^{(k)} \times R^{(k)}$, we make G a closed ind-subvariety of E. Then $G \hookrightarrow E$ is a closed embedding, and the uniqueness follows from Corollary 7. □

The group G equipped with the ind-variety structure of Proposition 6 is actually an affine ind-group. Indeed, by construction, G is an affine ind-variety. We need only to show that $\mu : G \times G \to G,\ (\sigma, \tau) \mapsto \sigma \circ \tau^{-1}$, is a morphism of ind-varieties. For this, it suffices to show that for each $k \geq 0$, there is $m = m(k) \geq 0$ such that $\mu((G \times G)^{(k)}) \subseteq G^{(m)}$. But since G admits an amalgamated decomposition, a standard inductive argument (see, e.g, [K1, Lemma 4.1]) shows that $\deg(\tau^{-1}) \leq \deg(\tau)$ for all $\tau \in G$. This implies that $\mu((G \times G)^{(k)}) \subseteq G^{(k^2)}$ for all k.

Remark. The full automorphism group $\tilde{G} := \mathrm{Aut}(R)$ of the algebra R has also a natural structure of an ind-group. In fact, by Dicks' Theorem, $\tilde{G}$ is the preimage of the subset $\{\lambda w \in R \,:\, \lambda \in \mathbb{C}^*\}$ which is locally closed in the ind-topology of R. By [FuM, Lemma 2], $\tilde{G}$ is then locally closed in E and hence has the structure of an ind-subvariety of E with induced filtration $\tilde{G}^{(k)} = \tilde{G} \cap E^{(k)}$.

We record two basic properties of the ind-group G which are similar to the properties of the Shafarevich ind-group $\mathrm{Aut}\,\mathbb{C}[x, y]$ (see [Sh1, Sh2]). First, recall (*cf.* [Sh2, Bl]) that an ind-variety X is *path connected* if for any $x_0, x_1 \in X$, there is an open set $U \subset \mathbb{A}^1_{\mathbb{C}}$ containing 0 and 1 and a morphism $f : U \to X$ such that $f(0) = x_0$ and $f(1) = x_1$.

Lemma 11. *A path connected ind-variety is connected.*

Proof. Indeed, a morphism of ind-varieties $f : U \to X$ is continuous. Hence, if X is the disjoint union of two proper closed subsets which intersect with $\mathrm{Im}(f)$, the preimages of these subsets under f must be non-empty, closed and disjoint in U. This contradicts the fact that U is connected in the Zariski topology. □

Theorem 13. *The group G is connected and hence irreducible.*

Proof. By Lemma 11, it suffices to show any element of G can be joined to the identity element $e \in G$ by a morphism $f : U \to G$. By Theorem 7, any $\sigma \in G$ can be written as a composition $\sigma = \Phi_{p_1}\Psi_{q_1} \ldots \Phi_{p_n}\Psi_{q_n}$ of transformations (11). Rescaling the polynomials p_i and q_i, we define

$$f : \mathbb{A}^1_{\mathbb{C}} \to G\ , \quad t \mapsto \Phi_{tp_1}\Psi_{tq_1} \ldots \Phi_{tp_n}\Psi_{tq_n}\ , \tag{41}$$

which is obviously a morphism of ind-varieties such that $f(1) = \sigma$ and $f(0) = e$. Hence G is connected. On the other hand, it is known that any connected ind-group is actually irreducible (see [Sh2, Prop. 3] and also [Ku, Lemma 4.2.5]). □

The next theorem is the analogue of [Sh1, Theorem 8].

Theorem 14. *Every finite-dimensional algebraic subgroup of G is conjugate to either a subgroup of A or a subgroup of B.*

Proof. The proof is essentially the same as in the classical case; we recall it for reader's convenience. By definition, an algebraic subgroup H of G is a closed subgroup which is again an ind-group with respect to the closed subvariety structure on $H^{(k)} = H \cap G^{(k)}$. In particular, each $H^{(k)}$ is closed in H and hence, when H is finite-dimensional, there is a $k \geq 0$ such that $H^{(k)} = H^{(k+1)} = \ldots = H$. It follows that $H \subset G^{(k)}$ for some k, which means that

$$\deg(\sigma) \leq k \quad \text{for all} \quad \sigma \in H \ . \tag{42}$$

On the other hand, by Theorem 7, every element $\sigma \in G$ has a reduced decomposition of the form

$$\sigma = a_1\, b_1\, \ldots\, a_l\, b_l\, a_{l+1}$$

where the $b_i \in B \setminus A$ for all $1 \leq i \leq l$ and $a_j \in A \setminus B$ for $2 \leq j \leq l$. The number l is independent of the choice of a decomposition and called the length of σ. As in the case of polynomial automorphisms (see [Wr, FM]), the length and the degree of σ are related by the formula $\deg(\sigma) = \deg(b_1)\deg(b_2)\ldots\deg(b_l)$, which shows that any subset of G, bounded in degree, is also bounded in length. Thus (42) implies that H is a subgroup of G bounded in length. A theorem of Serre (see [Se, Thm 4.3.8]) then implies that H is contained in a conjugate of A or B. $\square$

5.3. **The ind-algebraic structure on G_n.** We have seen in Section 3.5 that each G_n is a maximal subgroup of G. It therefore natural to expect that G_n is an algebraic subgroup of G. This follows formally from the next theorem.

Theorem 15. *The ind-group G acts algebraically on each space $\mathcal{C}_n$.*

Proof. Recall that the action of G on $\mathcal{C}_n$ is defined by (12). To see that this action is algebraic we first consider

$$G \times \mathcal{M}_n(\mathbb{C})^{\times 2} \xrightarrow{s \times \mathrm{Id}} G \times \mathcal{M}_n(\mathbb{C})^{\times 2} \xrightarrow{\iota \times \mathrm{Id}} E \times \mathcal{M}_n(\mathbb{C})^{\times 2} \xrightarrow{\mathrm{ev}} \mathcal{M}_n(\mathbb{C})^{\times 2}\ , \tag{43}$$

where $s : G \to G$ is the inverse map on G, $\iota : G \hookrightarrow E$ is the natural inclusion and ev is the evaluation map defined by $[(p, q),\, (X, Y)] \mapsto (p(X, Y),\, q(X, Y))$. Each arrow in (43) is a morphism of ind-varieties with respect to the product ind-variety structure on $E \times \mathcal{M}_n(\mathbb{C})^{\times 2}$; hence (43) defines an algebraic action of G on $\mathcal{M}_n(\mathbb{C})^{\times 2}$. This restricts to an action $G \times \tilde{\mathcal{C}}_n \to \tilde{\mathcal{C}}_n$, which is also algebraic since $\tilde{\mathcal{C}}_n$ is a closed subvariety of $\mathcal{M}_n(\mathbb{C})^{\times 2}$. Finally, as PGL_n acts freely on $\tilde{\mathcal{C}}_n$ and G commutes with PGL_n, the quotient map $G \times \tilde{\mathcal{C}}_n \to \tilde{\mathcal{C}}_n \twoheadrightarrow \mathcal{C}_n$ is algebraic and it induces an algebraic action $G \times \mathcal{C}_n \to \mathcal{C}_n$, which is precisely (12). $\square$

By definition, G_n is the fibre of the action map $p : G \to \mathcal{C}_n,\ \sigma \mapsto \sigma(X_0, Y_0)$, over the basepoint $(X_0, Y_0) \in \mathcal{C}_n$. By Theorem 15, this map is a morphism of ind-varieties: hence, G_n is a closed subgroup of G. We believe that the following is true.

Conjecture 3. *The groups G_n are connected and hence irreducible for all $n \geq 0$.*

In Section 4.3, we have explicitly described the structure of G_n as a discrete group for small n. Using this explicit description, we can easily prove

Proposition 7. *Conjecture 3 is true for* $n = 0,\, 1,\, 2$.

Proof. For $n = 0$, this is just Theorem 13. The argument of Theorem 13 can be also extended to G_1 and G_2, since we know explicit generating sets for these groups. Precisely, as shown in Section 4.3.2, G_1 is generated by Φ_p and Ψ_q with $p \in \mathbb{C}[x]$ and $q \in \mathbb{C}[y]$ satisfying $p(0) = q(0) = 0$. Hence, for any $\sigma \in G_1$, the morphism (41) constructed in the proof of Theorem 13 has its image in G_1, which, by Lemma 11, implies that G_1 is connected. Similarly, by (37), G_2 is generated by its subgroups

$G_{2,x}$, $G_{2,y}$, $G_{2,y}^{(1)}$ and T, each of which is path connected. For example, every $\sigma \in G_{2,y}^{(1)}$ has the form $\Phi_{-x}\Psi_q\Phi_x$, with $q \in \mathbb{C}[y]$ satisfying $q(\pm 1) = 0$. The last condition is preserved under rescaling $t \mapsto tq$. Hence, we can join σ to e within $G_{2,y}^{(1)}$ by the algebraic curve $f : t \mapsto \Phi_{-x}\Psi_{tq}\Phi_x$. This shows that G_2 is connected. □

Unfortunately, for $n \geq 3$, the Bass-Serre decomposition of G_n is too complicated, and its factors (generating subgroups of G_n) are much harder to analyze. In general, one might try a different approach using basic topology[7]. First, observe that, since G acts transitively on $\mathcal{C}_n$ and $\dim \mathcal{C}_n < \infty$, the action map $p : G \to \mathcal{C}_n$ restricts to a surjective morphism of affine varieties $p_k : G^{(k)} \to \mathcal{C}_n$ for $k \gg 0$. Thus there is a fibration

$$\begin{tikzcd} G_n^{(k)} \arrow[r, hook] \arrow[d] & G^{(k)} \arrow[d, "p_k"] \\ * \arrow[r] & \mathcal{C}_n \end{tikzcd} \tag{44}$$

Since $G^{(k)}$ and $\mathcal{C}_n$ are algebraic varieties over $\mathbb{C}$, we can equip them with classical topology: we write $G^{(k)}(\mathbb{C})$ and $\mathcal{C}_n(\mathbb{C})$ for the corresponding complex analytic varieties. The key question then is

Question. Is (44) locally trivial in classical topology for $k \gg 0$?

Assume (for a moment) that the answer is 'yes'. Then, for $k \gg 0$, there is an exact sequence associated to (44):

$$\ldots \to \pi_1\,(G^{(k)}(\mathbb{C}),\, \tilde{x}_0) \xrightarrow{(p_k)_*} \pi_1\,(\mathcal{C}_n(\mathbb{C}),\, x_0) \to \pi_0\,[G_n^{(k)}(\mathbb{C})] \to 0\ , \tag{45}$$

where $\pi_0\,[G_n^{(k)}(\mathbb{C})]$ is the set of connected components of $G_n^{(k)}(\mathbb{C})$. Now, it is known that $\mathcal{C}_n(\mathbb{C})$ is homeomorphic to the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ and hence is simply connected. It follows from (45) that $G_n^{(k)}(\mathbb{C})$ is connected and hence $G_n^{(k)}$ is connected in Zariski topology. If this holds for *all* $k \gg 0$, then by [Sh2, Prop. 2] (see also [K2, Prop. 2.4]), the ind-variety G_n must be connected.

5.4. **A nonreduced ind-scheme structure on G.** In this section, we define another ind-algebraic structure on the group G. Although, strictly speaking, this structure does not obey Shafarevich's definition, in some respects, it is more natural than the one introduced in Section 5.2.

Recall that, set-theoretically, G can be identified with the fibre over $w = [x, y]$ of the commutator map $c : E \to R$, see (40). As shown in Section 5.2, c is a morphism of two affine ind-varieties equipped with canonical filtrations. Let $c_k : E^{(k)} \to R^{(2k)}$ denote the restriction of c to the corresponding filtration components; by construction, $E^{(k)}$ and $R^{(2k)}$ are finite-dimensional vector spaces and c_k is a polynomial map. Now, for each $k \geq 0$, we define $\mathcal{G}^{(k)}$ to be the *scheme-theoretic* fibre of c_k over the closed point $w \in R^{(2k)}$: that is,

$$\mathcal{G}^{(k)} := \mathrm{Spec}\ \mathbb{C}[E^{(k)}]/c_k^*(\mathfrak{m}_w)$$

where $\mathfrak{m}_w \subset \mathbb{C}[R^{(2k)}]$ is the maximal ideal corresponding to w. Clearly, for all k, we have closed embeddings of affine schemes

$$\ldots \hookrightarrow \mathcal{G}^{(k-1)} \hookrightarrow \mathcal{G}^{(k)} \hookrightarrow \mathcal{G}^{(k+1)} \hookrightarrow \ldots \tag{46}$$

[7]We thank P. Etingof for suggesting us this idea.

induced by the natural inclusions $E^{(k-1)} \subset E^{(k)} \subset E^{(k+1)}$. Moreover, if we identify $G^{(k)}$ (the set-theoretic fibre of c_k) with $\operatorname{Spec}\mathbb{C}[\mathcal{G}^{(k)}]_{\rm red}$, we get the commutative diagram of affine schemes

$$\begin{tikzcd} \ldots \arrow[r, hook] & G^{(k-1)} \arrow[r, hook] \arrow[d] & G^{(k)} \arrow[r, hook] \arrow[d] & G^{(k+1)} \arrow[r, hook] \arrow[d] & \ldots \\ \ldots \arrow[r, hook] & \mathcal{G}^{(k-1)} \arrow[r, hook] & \mathcal{G}^{(k)} \arrow[r, hook] & \mathcal{G}^{(k+1)} \arrow[r, hook] & \ldots \end{tikzcd} \tag{47}$$

with vertical arrows corresponding to the algebra projections $\mathbb{C}[\mathcal{G}^{(k)}] \twoheadrightarrow \mathbb{C}[\mathcal{G}^{(k)}]_{\rm red}$.

The filtration (46) defines on G the structure of an affine ind-scheme[8], which we denote by $\mathcal{G}$. The diagram (47) gives a canonical morphism of affine ind-schemes

$$i : G \to \mathcal{G} \tag{48}$$

that reduces to the identity map on the underlying sets. We will prove

Proposition 8. *The map i is* not *an isomorphism of affine ind-schemes.*

We begin with general remarks on tangent spaces. Recall that, if X is an affine $\mathbb{C}$-scheme and $x \in X$ a closed point, the (Zariski) tangent space to X at x is defined by $T_xX := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, or equivalently

$$T_xX = \operatorname{Der}(\mathbb{C}[X], \mathbb{C}_x) ,$$

where $\mathbb{C}_x = \mathbb{C}$ is viewed as a $\mathbb{C}[X]$-module via the algebra map $\mathbb{C}[X] \to \mathbb{C}$ corresponding to x. A morphism $f : X \to Y$ of affine schemes defines an algebra map $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$, which in turn, induces the linear map

$$df_x : T_xX \to T_yY , \quad \partial \mapsto \partial \circ f^* ,$$

called the differential of f at x. The kernel of df_x is canonically isomorphic to the tangent space of $\mathcal{Z} := \operatorname{Spec}\mathbb{C}[X]/f^*(\mathfrak{m}_y)$, the scheme-theoretic fibre of f over $y = f(x)$, and we identify

$$T_x\mathcal{Z} = \operatorname{Ker}(df_x) .$$

If X, Y are varieties and $Z := \mathcal{Z}_{\rm red} = f^{-1}(y)$ is the (set-theoretic) fibre of f, then there is a canonical map $i : Z \hookrightarrow \mathcal{Z}$, which, for all $x \in Z$, induces an inclusion

$$di_x : T_xZ \hookrightarrow T_x\mathcal{Z} = \operatorname{Ker}(df_x) . \tag{49}$$

In particular, if $\mathcal{Z}$ is reduced (i.e., i is an isomorphism), then $T_xZ = \operatorname{Ker}(df_x)$.

Now, let X be an affine ind-variety with filtration $\{X^{(k)}\}$. For any $x \in X$, there is $k_0 \geq 0$ such that $x \in X^{(k)}$ for all $k \geq k_0$. Hence, the filtration embeddings $X^{(k)} \hookrightarrow X^{(k+1)}$ induce the sequential direct system of vector spaces

$$T_xX^{(k)} \to T_xX^{(k+1)} \to T_xX^{(k+2)} \to \ldots ,$$

and the corresponding direct limit $T_xX := \varinjlim_k T_xX^{(k)}$ is called the tangent space to X at x (*cf.* [Ku, 4.1.4]). If $X = V$ is an ind-vector space filtered by finite-dimensional subspaces $V^{(k)}$ (see Example 2), then, for each $k \geq 0$, we can identify $T_xV^{(k)} = V^{(k)}$, using the canonical isomorphism

$$V^{(k)} \xrightarrow{\sim} T_xV^{(k)} , \quad v \mapsto \partial_{v,x} , \tag{50}$$

[8] By an affine ind-scheme we mean a countable inductive limit of closed embeddings in the category of affine $\mathbb{C}$-schemes (*cf.* [K2, K3]). Clearly, any affine ind-variety in the sense of Section 5.1 is an example of an affine ind-scheme.

where $\partial_{v,x} \in \mathrm{Der}(\mathbb{C}[V], \mathbb{C})$ is defined by $\partial_{v,x} F = \partial_v F(x) := (d/dt)[F(x+vt)]_{t=0}$. With these identifications, the natural inclusions $T_x V^{(k)} = V^{(k)} \hookrightarrow V$ induce an injective linear map $T_x V = \varinjlim_k T_x V^{(k)} \hookrightarrow V$, which is actually an isomorphism, since $\varinjlim_k V^{(k)} = \bigcup_{k \gg 0} V^{(k)} = V$. Thus, just as in the finite-dimensional case, we can identify $T_x V = V$ using (50).

Next, let $f : V \to W$ be a morphism of ind-varieties, each of which is an ind-vector space of countable dimension. Fix $w \in W$ and let $Z := f^{-1}(w)$ be the (set-theoretic) fibre of f over w. Assume that $Z \neq \varnothing$ and put the induced topology on Z, i.e. $Z^{(k)} = Z \cap V^{(k)}$. Then, there is $k_0 \geq 0$ such that $w \in \mathrm{Im}(f_k)$ for all $k \geq k_0$, and we obviously have $Z^{(k)} = f_k^{-1}(w)$, where $f_k : V^{(k)} \to W^{(m)}$ is the restriction of f to $V^{(k)}$. Let $\mathcal{Z}^{(k)} := \mathrm{Spec}\, \mathbb{C}[V^{(k)}]/f_k^*(\mathfrak{m}_w)$ be the scheme-theoretic fiber of f_k over $w \in W^{(k)}$, and let $i_k : Z^{(k)} \hookrightarrow \mathcal{Z}^{(k)}$ be the canonical maps. Then, by (49), we have

$$(di_k)_x : T_x Z^{(k)} \hookrightarrow T_x \mathcal{Z}^{(k)} = \mathrm{Ker}\,(df_k)_x$$

for all $k \geq k_0$. Hence

$$T_x Z := \varinjlim T_x Z^{(k)} \hookrightarrow \varinjlim \mathrm{Ker}\,(df_k)_x = T_x \mathcal{Z} \ .$$

On the other hand, since $\varinjlim$ preserves exact sequences, the exactness of

$$0 \to \mathrm{Ker}\,(df_k)_x \to V^{(k)} \xrightarrow{(df_k)_x} W^{(m)}$$

implies $\varinjlim \mathrm{Ker}(df_k)_x = \mathrm{Ker}\,(df)_x = \{v \in V : \partial_v f(x) = 0\}$. Thus,

$$T_x \mathcal{Z} = \{v \in V : \partial_v f(x) = 0\} \ , \tag{51}$$

and the differential of $i : Z \to Z$ induces an inclusion $di_x : T_x Z \hookrightarrow T_x \mathcal{Z}$, which is an isomorphism whenever the fibres of f_k are reduced for all $k \gg 0$.

To prove Proposition 8 we apply the above remarks to the morphism $c : E \to R$. Under (39), the identity element of G corresponds to $e = (x, y) \in E$, and for any $\boldsymbol{v} = (u, v) \in R^2$, we have

$$\partial_{\boldsymbol{v}} c(e) = (d/dt)[c(e + t\boldsymbol{v})]_{t=0} = [x, v] + [u, y] \ .$$

Hence, by (51), we can identify

$$T_e \mathcal{G} = \{(u, v) \in R^2 \ : \ [x, v] + [u, y] = 0\} \ . \tag{52}$$

and the differential of (48) gives an embedding

$$di_e : T_e G \hookrightarrow T_e \mathcal{G} \ . \tag{53}$$

Proof of Proposition 8. It suffices to show that (53) is not surjective. We identify $T_e G$ with the image of di_e and show that $T_e G \neq T_e \mathcal{G}$. To this end we will use the canonical anti-involution on the algebra R defined by $x^\dagger = x$ and $y^\dagger = y$ and $(ab)^\dagger = b^\dagger a^\dagger$ for all $a, b \in R$. The elements of R invariant under $\dagger$ are called *palindromic*: we write $R^\dagger := \{a \in R \ : \ a^\dagger = a\}$. Extending $\dagger$ to $E = R^2$ by $(p, q)^\dagger := (p^\dagger, q^\dagger)$, we get an (auto)morphism of the ind-variety E. Now, it is known (see [SY, Corollary 1.5]) that the map $\dagger : E \to E$ restricts to the identity map on G and hence induces the identity map on $T_e G$. This means that $T_e G \subseteq (T_e \mathcal{G})^\dagger$, where $(T_e \mathcal{G})^\dagger := \{(u, v) \in T_e \mathcal{G} \ : \ u^\dagger = u \ , \ v^\dagger = v\}$. Thus it suffices to show that $(T_e \mathcal{G})^\dagger \neq T_e \mathcal{G}$. This can be verified directly: for example, take in R the following elements

$$a = xy^2x^2 + x^2yxy + yx^2yx \ , \quad b = yx^2y^2 + y^2xyx + xy^2xy$$

and define $u := a - a^\dagger$ and $v := b - b^\dagger$, so that $u^\dagger = -u$ and $v^\dagger = -v$. Then obviously $(u,v) \notin (T_e\mathcal{G})^\dagger$, but a trivial calculation shows that $(u,v) \in T_e\mathcal{G}$. $\square$

Remarks. 1. The proof of Proposition 8 shows that the schemes $\mathcal{G}^{(k)}$ are actually nonreduced for $k \geq 5$.

2. There is an intrinsic Lie algebra structure on $T_e\mathcal{G}$ coming from the fact that $\mathcal{G}$ is an ind-group (*cf.* [Sh2], [Ku, Sect. 4.2]). With identification (52), the Lie bracket on $T_e\mathcal{G}$ can be described as follows. Let $\mathrm{Der}(R)$ be the Lie algebra of linear derivations of R, and let $\mathrm{Der}_w(R)$ be the subalgebra of $\mathrm{Der}(R)$ consisting of derivations that vanish at w. Identify $\mathrm{Der}(R) = R^2$ via the evaluation map $\delta \mapsto (\delta(x), \delta(y))$. Then $T_e\mathcal{G} \subset R^2$ corresponds precisely to $\mathrm{Der}_w(R)$ and the Lie bracket on $T_e\mathcal{G}$ corresponds to the commutator bracket on $\mathrm{Der}_w(R)$. Thus, as was originally suggested in [BW], there is an isomorphism of Lie algebras $\mathrm{Lie}(\mathcal{G}) \cong \mathrm{Der}_w(R)$.

3. Proposition 8 shows that the Lie algebra $\mathrm{Lie}(G)$ of the ind-group G is a *proper* subalgebra of $\mathrm{Lie}(\mathcal{G})$. In fact, $\mathrm{Lie}(G) \subseteq \mathrm{Lie}(\mathcal{G})^\dagger \subsetneqq \mathrm{Lie}(\mathcal{G})$. We expect that $\mathrm{Lie}(G)$ is generated by the two abelian Lie subalgebras $\mathrm{Lie}(G_x)$ and $\mathrm{Lie}(G_y)$, which are spanned by the derivations $\{(x^k, 0)\}_{k\in\mathbb{N}}$ and $\{(0, y^m)\}_{m\in\mathbb{N}}$ (*cf.* [EG, Question 17.10]).

4. There is an appealing description of the Lie algebra $\mathrm{Der}_w(R)$, due to Kontsevich [Ko] (see also [G] and [BL]). Specifically, $\mathrm{Der}_w(R)$ can be identified with $\bar{\mathfrak{L}} := R/([R,R] + \mathbb{C})$, the space of cyclic words in the variables x and y. The Lie bracket on $\mathrm{Der}_w(R)$ corresponds to a Poisson bracket on $\bar{\mathfrak{L}}$ defined in terms of cyclic derivatives (see [Ko, Sect. 6]). Note that the Lie algebra $\bar{\mathfrak{L}}$ has an obvious one-dimensional central extension: $\mathfrak{L} := R/[R,R]$. By a theorem of Ginzburg [G], the varieties $\mathcal{C}_n$ can be naturally embedded in $\mathfrak{L}^*$ as coadjoint orbits, and thus can be identified with coadjoint orbits of a central extension of $\mathcal{G}$. Conjectures 1 and 2 in Section 3.6 suggest that Ginzburg's theorem extends to all configuration spaces $\mathcal{C}_n^{[k]}$ and their products $\mathcal{C}_{n_1}^{[k_1]} \times \mathcal{C}_{n_2}^{[k_2]} \times \ldots \times \mathcal{C}_{n_m}^{[k_m]}$ (with $n_i \neq n_j$).

6. Borel Subgroups

In this section, we will study the Borel subgroups of G_n and prove our main classification theorems stated in the Introduction. Recall that the natural action of the group G on $\mathbb{C}^2$ is faithful (see Proposition 1 and remark thereafter). Using this action, we will identify G as a *discrete* group with a subgroup of polynomial automorphisms in $\mathrm{Aut}(\mathbb{C}^2)$: in other words, we will think of the elements of G (and hence G_n) as automorphisms of $\mathbb{C}^2$. This will allow us to apply the results of [FM] and [L]. On the other hand, to define Borel subgroups we will regard G and G_n as *topological* groups with (reduced) ind-Zariski topology introduced in Section 5.

6.1. **Friedland-Milnor-Lamy classification.** By [FM], the elements of G can be divided into two separate classes according to their dynamical properties as automorphisms of $\mathbb{C}^2$: every $g \in G$ is conjugate to either an element of B or a composition of generalized Hénon automorphisms of the form:

$$\sigma\, g\, \sigma^{-1} = g_1\, g_2\, \ldots\, g_m\ ,$$

where $g_i = (y, x + q_i(y))$ with polynomials $q_i(y) \in \mathbb{C}[y]$ of degree ≥ 2. We say that g is of *elementary* or *Hénon* type, respectively. A subgroup $H \subseteq G$ is called *elementary* if each element of H is of elementary type.

It is convenient to reformulate this classification in terms of the action of G on the standard tree $\mathcal{T}$ associated to the amalgam $G = A *_U B$. By definition, the vertices $V(\mathcal{T})$ of $\mathcal{T}$ are the left cosets $G/A \sqcup G/B$, while the set of edges is $E(\mathcal{T}) = G/U$. The group G acts on $\mathcal{T}$ by left translations. Notice that if $g \in G$ fixes two vertices in $\mathcal{T}$, then it also fixes all the vertices linking these two vertices; thus, for each $g \in G$, we may define a subtree $\mathtt{Fix}(g) \subseteq \mathcal{T}$ fixed by g. More generally, if H is a subgroup of G, following [L], we put $\mathtt{Fix}(H) := \cap_{g\in H} \mathtt{Fix}(g)$. It is easy to see that $\mathtt{Fix}(g)$ is non-empty iff g is elementary, and $\mathtt{Fix}(g) = \varnothing$ iff g is of Hénon type. In the latter case, following [Se], we may define the *geodesic* of g to be the set of vertices of $\mathcal{T}$ that realizes the infimum $\inf_{p\in V(\mathcal{T})} \mathrm{dist}(p, g(p))$, where $\mathrm{dist}(p, q)$ is the number of edges of the shortest path joining the vertices p and q in $\mathcal{T}$.

The following theorem is a consequence of the main result of S. Lamy.

Theorem 16 ([L]). *Let H be a subgroup of G. Then, one and only one of the following possibilities occurs:*

(I) *H is an elementary subgroup conjugate to a subgroup of A or B.*

(II) *H is an elementary subgroup which is not conjugate to a subgroup of A or B. Then H is countable and abelian.*

(III) *H contains elements of Hénon type, and all such elements in H share the same geodesic. Then H is solvable and contains a subgroup of finite index isomorphic to $\mathbb{Z}$.*

(IV) *H contains two elements of Hénon type with distinct geodesics. Then H contains a free subgroup on two generators.*

Remarks. 1. Theorem 16 is essentially Théorème 2.4 of [L], except that this last paper is concerned with subgroups of the full automorphism group $\mathrm{Aut}(\mathbb{C}^2)$. As a subgroup of $\mathrm{Aut}(\mathbb{C}^2)$, G coincides with the kernel of the Jacobian map $\mathrm{Jac} : \mathrm{Aut}(\mathbb{C}^2) \to \mathbb{C}^*$, which splits and gives an identification $\mathrm{Aut}(\mathbb{C}^2) \cong G \rtimes \mathbb{C}^*$. Using this we can easily deduce Theorem 16 from [L, Théorème 2.4]. Indeed, if H is an elementary subgroup of G, then (by definition) it is elementary in $\mathrm{Aut}(\mathbb{C}^2)$ and hence is either of type I or type II in that group. If H is of type II in $\mathrm{Aut}(\mathbb{C}^2)$ then it is automatically of type II in G. If H is of type I in $\mathrm{Aut}(\mathbb{C}^2)$, then, by [L, Théorème 2.4], it can be conjugate to a subgroup $\tilde{H}$ of $\mathrm{Aut}(\mathbb{C}^2)$, which is either in $A \rtimes \mathbb{C}^*$ or $B \rtimes \mathbb{C}^*$. But $\mathrm{Jac}(\tilde{H}) = \mathrm{Jac}(H) = 1$, hence $\tilde{H} \subset G$, and since $\mathrm{Aut}(\mathbb{C}^2) \cong G \rtimes \mathbb{C}^*$. we can conjugate H to $\tilde{H}$ within G. For types III and IV, the implication Théorème 2.4 $\Rightarrow$ Theorem 16 is automatic.

2. We have added to [L, Théorème 2.4] that any subgroup H of type III contains a finite index subgroup isomorphic to $\mathbb{Z}$. Indeed, by [L, Prop. 4.10], all subgroups of G satisfying property (III) for a *fixed* geodesic generate a unique largest subgroup, which contains a copy of $\mathbb{Z}$ as a subgroup of finite index. The last condition means that $H \cap \mathbb{Z}$ is a subgroup of finite index in H; hence H either contains $H \cap \mathbb{Z} \cong \mathbb{Z}$ as a subgroup of finite index or is finite (if $H \cap \mathbb{Z} = \{1\}$). It remains to note that the last possibility does not occur, since, by [Se, Theorem 8, Sect. I.4.3], any finite subgroup of G is conjugate to a subgroup of A or B and hence is of type I.

As a consequence of Theorem 16, the following *Tits alternative* holds for G (*cf.* [L], Corollary 2.5): every subgroup of G contains either a solvable subgroup of finite index or a non-abelian free group.

6.2. **Solvable subgroups of G.** We begin with the following observation which may be of independent interest.

Lemma 12. *For $g \in G$, one and only one of the following possibilities occurs:*

(a) *g is elementary, and $\langle g \rangle \cong \mathbb{Z}$,*
(b) *g is elementary and $\langle g \rangle \cong \mathbb{Z}_n$ for some $n \geq 1$,*
(c) *g is Hénon type and $\langle g \rangle \cong \mathbb{Z}$.*

Moreover, $\langle g \rangle$ is a closed subgroup of G if and only if it is as in (b) *or* (c).

Proof. By the Friedland-Milnor classification, any element of G is either elementary (i.e. conjugate to an element of B) or of Hénon type.

Suppose that g is elementary. Then we may assume that g is contained in B. If g has finite order, $\langle g \rangle \cong \mathbb{Z}_n$ for some $n \geq 1$. Since $\langle g \rangle$ is finite, it is a closed subgroup of G. If g has infinite order then $\langle g \rangle \cong \mathbb{Z}$. Moreover $\langle g \rangle \subset G^{(k)}$, for some k, where $G^{(k)}$ is k-th filtration component of G. Since $\|\langle g \rangle\|$ is countable, it cannot be closed in $G^{(k)}$.

Suppose g is of Hénon type. For a Hénon automorphism, the sequence $\{\deg(g^k)\}_{k=1}^{\infty}$ is strictly increasing, and we have $\lim_{k\to\infty} \deg(g^k) = \infty$. For any $n > 0$, $G^{(n)} \cap \langle g \rangle$ is finite and hence closed. Thus $\langle g \rangle$ is equipped with an increasing filtration of closed sets therefore it is an ind-subgroup of G. □

Proposition 9. *Let H be a subgroup of G with either of the following properties:*

(S1) *H is a solvable group without a proper subgroup of finite index,*
(S2) *H is a connected solvable group.*

Then H cannot be of type III (in the nomenclature of Theorem 16).

Proof. Suppose H is a type III subgroup. Then H is a subgroup of the group K explicitly described in [L, Proposition 4.10]. By the proof of this proposition, K has a finite index subgroup generated by a Hénon type element. We denote this subgroup by K_1. Since $H \subseteq K$, we have $H/(H \cap K_1) \subseteq K/K_1$ and $H/(H \cap K_1)$ is finite. This shows that H with property (S1) cannot be of type III, since $H \cap K_1$ is a subgroup of finite index in H.

Since K_1 is closed in G, by Lemma 12(c), $H \cap K_1$ is closed in H. Therefore $H = \bigcup_{i=1}^{n} g_i(H \cap K_1)$ is the disjoint union of closed subsets. Since H is connected, we must have $H \cap K_1 = H$, hence $H \subseteq K_1$. It follows that either $H = \langle g \rangle$ for some $g \in K_1$ or $H = 1$. One can easily see that $H = \langle g \rangle$ cannot be connected: $H_1 = \langle g^2 \rangle$ is its closed subgroup of index 2, hence $H = gH_1 \cup H_1$ is the disjoint union of closed subsets. Therefore $H = 1$. This proves (S2). □

6.3. **Borel subgroups of G.** Recall that a subgroup of a topological group is called *Borel* if it is connected, solvable and maximal among all connected solvable subgroups. For basic properties of Borel subgroups we refer to [Bo1, § 11]. We only note that any Borel subgroup is necessarily a closed subgroup.

We begin with the following proposition which establishes the main properties of the subgroup of triangular automorphisms in G.

Proposition 10. *Let B be the subgroup of G defined by* (9)*. Then*

(a) *B is a solvable group of derived length 3.*
(b) $\mathtt{Fix}(B) = \{1 \cdot B\}$ *consists of a single vertex.*
(c) *$N_G(B) = B$.*
(d) *B is a connected subgroup of G.*

(e) *B is a maximal solvable subgroup of G.*

In particular, B is a Borel subgroup of G.

Proof. (a) One can easily compute the derived series of B, which is given by

$$B^{(1)} = \{(x+p(y), y+f) \,|\, f \in \mathbb{C}\,,\, p(y) \in \mathbb{C}[y]\}\,, \quad B/B^{(1)} \cong \mathbb{C}^*$$

$$B^{(2)} = \{(x+p(y), y) \,|\, p(y) \in \mathbb{C}[y]\}\,, \quad B^{(1)}/B^{(2)} \cong \mathbb{C}$$

(b) It is clear that $\mathtt{Fix}(B)$ contains $\{1 \cdot B\}$. Now, by [L, Proposition 3.3], there is an element $f \in B$ such that $\mathtt{Fix}(f) = \{1 \cdot B\}$. Hence $\mathtt{Fix}(B) = \{1 \cdot B\}$.

(c) Let $g \in N_G(B)$, i.e. $g^{-1}Bg \subseteq B$. Then $B \subseteq gBg^{-1}$. Hence B must also fix the vertex $g \cdot B$. By part (b), $g \cdot B = 1 \cdot B$ and $g \in B$.

(d) By Lemma 11, it suffices to show that B is path connected. Let $b = (tx + p(y), t^{-1}y + f)$ be an arbitrary element in B. Consider $b_s = (tx + s\,p(y), t^{-1}y + s\,f) \in B$ for $s \in \mathbb{C}$. We have $b_0 = (tx, t^{-1}y)$ and $b_1 = b$. Thus, every element of B is connected to the subgroup $T = \{(tx, t^{-1}y) \,|\, t \in \mathbb{C}^*\}$. On the other hand, T is path connected, hence B is path connected as well.

(e) Suppose B is contained in a solvable subgroup $H \subset G$. Then, H is a solvable group of length at least 3. Then, by Theorem 16, it is either of type I or type III. By Proposition 9, it can be only of type I: i.e., it is conjugate to a subgroup of either A or B. Suppose that there is $g \in G$ such that $g^{-1}Bg \subseteq g^{-1}Hg \subset A$. Then $B \subset gAg^{-1}$. This implies that B fixes the vertex $g \cdot A$, which contradicts part (b). Suppose that there is $g \in G$ such that $g^{-1}Bg \subseteq g^{-1}Hg \subset B$. Once again, we can conclude that B fixes $g \cdot B$ and hence $g \in B$ by (b). It follows that $B = H$ and hence B is maximal solvable. $\square$

Now, we can prove Theorem 3 from the Introduction.

Proof of Theorem 3. Let H be a Borel subgroup of G. Then, by classification of Theorem 16, H can only be a subgroup of type I. Indeed, it is obvious that H cannot be of type IV (since it is solvable); it cannot be of type III (by Proposition 9), and it cannot be of type II, since, by [L, Proposition 3.12], any type II subgroup of G is given by a countable union of finite cyclic groups and hence is totally disconnected in the ind-topology of G (*cf.* [Ku, 4.1.3(5)]). Thus H is conjugate to either a subgroup of A or a subgroup of B. In the first case, it must be a Borel subgroup of A. Since A is a connected algebraic subgroup of G, by the classical Borel Theorem, all Borel subgroups of A are conjugate to each other. Since U is a Borel in A, H must be conjugate to U. This obviously contradicts the maximality of H since U is properly contained in B. Hence H is conjugate to a subgroup of B; by maximality, it must then be conjugate to B. $\square$

The next lemma is elementary: we recall it for reader's convenience (the proof can be found, for example, in [H]).

Lemma 13. *Let G be an abstract group.*

(a) *If G has a proper subgroup of finite index then G has a proper normal subgroup of finite index.*

(b) *If G has no proper subgroup of finite index then any homomorphic image of G has no proper subgroup of finite index.*

(c) *If G is solvable and has no proper finite index subgroup, then it is infinitely generated.*

Using Lemma 13, we can now prove

Lemma 14. *The group B contains no proper subgroups of finite index.*

Proof. Suppose H is a proper finite index subgroup of B. By Lemma 13(a), we may assume that H is normal. Consider the quotient map $p_1 : B \twoheadrightarrow B/B^{(1)} \cong \mathbb{C}^*$. Then, the image of H under p_1 is a finite index subgroup of $\mathbb{C}^*$. But $\mathbb{C}^*$ has no proper finite index subgroups. Hence $p_1(H) = B/B^{(1)}$ and therefore $B = B^{(1)}H$. Now, let $H_1 := B^{(1)} \cap H$. Then

$$\frac{B}{H} = \frac{B^{(1)}H}{H} \cong \frac{B^{(1)}}{H_1}\,. \tag{54}$$

This implies that H_1 is a finite index subgroup of $B^{(1)}$. Next, we consider $p_2 : B^{(1)} \twoheadrightarrow B^{(1)}/B^{(2)} \cong \mathbb{C}$. Again, $p_2(H_1)$ is a finite index subgroup of $\mathbb{C}$. Since $\mathbb{C}$ has no proper finite index subgroups, we conclude $p_2(H_1) = B^{(1)}/B^{(2)}$. Hence $B^{(1)} = B^{(2)}H_1$, and we have

$$\frac{B^{(1)}}{H_1} \cong \frac{B^{(2)}}{B^{(2)} \cap H_1}\,. \tag{55}$$

Thus $B^{(2)}\cap H_1$ is a finite index subgroup $B^{(2)}$. On the other hand $B^{(2)} \cong \mathbb{C}[y]$ which has no proper finite index subgroups. Hence $B^{(2)} \cap H_1 = B^{(2)}$, which implies that $B^{(2)}$ is a subgroup of H_1. Next, since $B^{(1)} = B^{(2)}H_1$, we have $B^{(1)} = H_1 = B^{(1)}\cap H$. From this last equality we see that $B^{(1)} \subseteq H$. Finally, from $B = B^{(1)}H$ we get $H = B$. This contradicts the properness of H. $\square$

Before characterizing the Borel subgroups of G, we recall a classical characterization of solvable subgroups of $\mathrm{GL}_n(\mathbb{C})$ due to A. I. Maltsev. Maltsev's theorem can be viewed as a generalization of the Lie-Kolchin Theorem (*cf.* [Sp, 6.3.1]): for its proof we refer to [LR, Theorem 3.1.6].

Theorem 17 (Maltsev)**.** *Let Γ be any solvable subgroup of $\mathrm{GL}_n(\mathbb{C})$. Then Γ has a finite index normal subgroup which is conjugate to a subgroup of upper triangular matrices.*

We are now in position to prove Steinberg's Theorem for the group G.

Theorem 18. *Let H be a non-abelian subgroup of G. Then H is Borel iff*

(B1) *H is a maximal solvable subgroup of G,*
(B2) *H contains no proper subgroups of finite index.*

Proof. ($\Rightarrow$) Suppose H is Borel. Then, by Theorem 3, H is conjugate to B. Hence, by Proposition 10 and Lemma 14, H satisfies $(B1)$ and $(B2)$ respectively.

($\Leftarrow$) Let H be a non-abelian subgroup of G satisfying (B1) and (B2). Then, by Theorem 16, it is either of type I or type III. By Proposition 9, it cannot be of type III. Therefore, it is conjugate to a subgroup of A or B. Suppose that H is conjugate to a subgroup of A. The image of composition $g^{-1}Hg \to A \to \mathrm{SL}_2(\mathbb{C})$ is then a solvable subgroup of $\mathrm{SL}_2(\mathbb{C})$. We denote this group by S. By Theorem 17, S has a finite index normal subgroup T, which is a subgroup of upper triangular matrices in $\mathrm{SL}_2(\mathbb{C})$. By Lemma 13(b), the group S, being the image of H, contains no proper subgroups of finite index. Thus, $S = T$ and H is conjugate to a subgroup of U, which is a proper solvable subgroup of B. This contradicts the assumption that H is a maximal solvable subgroup of G. Hence, H can be only conjugate to a subgroup of B. Since H is maximal solvable, it must be conjugate to B itself. $\square$

Theorem 18 is the special case of Theorem 6 corresponding to $n = 0$. We now turn to the general case.

6.4. **Borel subgroups of G_n.** We begin with some technical lemmas. First, recall that, for any element $g = (P, Q) \in G$, we defined its degree in G by

$$\deg(g) := \max\{\deg(P),\, \deg(Q)\}\ ,$$

where $\deg(P)$ and $\deg(Q)$ are the degrees of $P = P(x, y)$ and $Q = Q(x, y)$ in the free algebra $\mathbb{C}\langle x, y\rangle$ (*cf.* Section 5.2). It is easy to see that $\deg(g)$ thus defined coincides with the degree of g viewed as an automorphism of $\mathbb{C}^2$.

Lemma 15. *Let H be a subgroup of B with the property that for any $N > 0$, there is $h \in H$ such that $\deg(h) > N$. If $H \subseteq g^{-1}B\,g \cap B$ for some $g \in G$, then $g^{-1}B\,g \cap B = B$ and $g \in B$.*

Proof. If $g \in B$, then $g^{-1}B\,g = B$ and therefore $g^{-1}\,B\,g\,\cap\,B = B$. Assume now that $g \in G\backslash B$. Then we can write $g = w_0 w_1\,\ldots\,w_l$, where $w_0 \in U$ and $\{w_1, \ldots, w_l\}$ are representatives of some cosets in A/U or B/U. Without loss of generality, we may assume that $w_0 = 1$ and w_1 is a coset representative from A/U. Then

$$g^{-1}B\,g\,\cap\,B = (w_l^{-1}\ldots w_2^{-1}\,w_1^{-1}\,U\,w_1\,w_2\ldots w_l)\,\cap\,B\ , \tag{56}$$

since $g^{-1}(B\backslash U)\,g$ consists of words of length $2l+1$ and $g^{-1}(B\backslash U)\,g\,\cap\,B = \varnothing$. Let $\deg(g) = n$. Then, by [K1, Lemma 4.1], $\deg(g^{-1}) \le n$, and the degrees of all elements in (56) are at most n^2. This contradicts the assumption that (56) contains H whose elements have arbitrary large degrees. $\square$

Now, for $g \in G$, we define $B_g := g^{-1}B\,g\,\cap\,G_n$. Clearly, B_g is a subgroup of G_n that depends only on the right coset of $g \in G$ (mod B). We write $V_n(B) := \{B_g\}_{g\in B}$ for the set of all such subgroups of G_n and note that G_n acts on $V_n(B)$ by conjugation.

Lemma 16. *The assignment $g \mapsto B_g$ induces a bijection*

$$\eta:\ B\backslash G \overset{\sim}{\to} V_n(B)\ ,$$

which is equivariant under the (*right*) *action of G_n.*

Proof. It is clear that the map η is well defined and surjective. We need only to prove that η is injective. Suppose that $g_1^{-1}B\,g_1\,\cap\,G_n = g_2^{-1}B\,g_2\,\cap\,G_n$ for some $g_1,\, g_2, \in G$. Then

$$g_2\,g_1^{-1}\,B\,g_1\,g_2^{-1}\,\cap\,g_2\,G_n\,g_2^{-1} = B\,\cap\,g_2\,G_n\,g_2^{-1}\ ,$$

which implies $B\,\cap\,g_2\,G_n\,g_2^{-1} \subseteq g_2\,g_1^{-1}\,B\,g_1\,g_2^{-1}\,\cap\,B$. Now, observe that $B\,\cap\,g_2\,G_n\,g_2^{-1} = \mathrm{Stab}_B[g_2\cdot(X_0, Y_0)]$. Hence $H := B\,\cap\,g_2\,G_n\,g_2^{-1}$ satisfies the assumptions of Lemma 15, and we conclude: $g_2\,g_1^{-1}\,B\,g_1\,g_2^{-1} = B$ and $g_1\,g_2^{-1} \in B$. It follows that $Bg_1 = Bg_2$. To see the equivariance of η, for $h \in G_n$, we compute

$$B_{gh} := (gh)^{-1}B\,(gh)\,\cap\,G_n = h^{-1}(g^{-1}B\,g\,\cap\,G_n)\,h = h^{-1}B_g\,h\ .$$

$\square$

Dividing the map η of Lemma 16 by the action of G_n, we get

$$\mathcal{C}_n/B \overset{\sim}{\to} V_n(B)/\mathrm{Ad}\,G_n\ , \tag{57}$$

where we have identified $B\backslash G/G_n = \mathcal{C}_n/B$ via $B\,g\,G_n \leftrightarrow B\,g(X_0, Y_0)$.

Notice that $V_n(B)$ is the set of B-vertex groups of the graph Γ_n constructed in Section 4.2. The next lemma gives a simple description of all vertex groups of Γ_n.

Lemma 17. *If $n \geq 1$, then, for any $g \in G$, there is*

(1) $\tilde{g} \in Ag$ *such that* $g^{-1}\, A\, g \,\cap\, G_n = \tilde{g}^{-1}\, \mathrm{SL}_2(\mathbb{C})\, \tilde{g} \,\cap\, G_n$,
(2) $\tilde{g} \in Bg$ *such that* $g^{-1}\, B\, g \,\cap\, G_n = \tilde{g}^{-1}\, (T \ltimes G_y)\, \tilde{g} \,\cap\, G_n$.

In particular, every B-vertex group of Γ_n is a solvable subgroup of G_n of derived length ≤ 2.

Proof. This follows from the fact that each A- and B-orbit in $\mathcal{C}_n$ contains a point (X, Y) with $\mathrm{Tr}(X) = \mathrm{Tr}(Y) = 0$. Indeed, both A and B contain translations, so we can move (X, Y) to $(X - \frac{1}{n}\,\mathrm{Tr}(X)\, I,\ Y - \frac{1}{n}\,\mathrm{Tr}(Y)\, I)$ along the orbits. $\square$

Proposition 11. *Let $B_g \in V_n(B)$. Then*

(a) *B_g is a solvable group of derived length ≤ 2 .*
(b) $N_{G_n}(B_g) = B_g$.
(c) *B_g is a maximal solvable subgroup of G_n .*

Proof. (a) By Lemma 17, B_g is isomorphic to a subgroup of $T \ltimes G_y$. Since $T \ltimes G_y$ is solvable of derived length 2, B_g is solvable of derived length at most 2.

(b) follows from Lemma 16 and the (obvious) fact that $\mathrm{Stab}_{G_n}(Bg) = B_g$.

(c) Let H be a solvable subgroup of G_n containing B_g. Since H is uncountable, by Theorem 16, it can only be of type I: i.e, conjugate either to a subgroup of A or a subgroup of B. In the the first case, $B_g \subseteq H \subseteq h^{-1} A\, h \,\cap\, G_n$ for some $h \in G_n$. This is impossible, since B_g contains elements of arbitarary large degree. Hence, H can only be conjugate to a subgroup of B, i.e. $B_g \subseteq H \subseteq h^{-1} B\, h \,\cap\, G_n$, and therefore $h\, B_g\, h^{-1} \subseteq (gh^{-1})^{-1} B\, gh^{-1} \cap B$. By Lemma 15, this implies $gh^{-1} \in B$, whence the equality $B_g = B_h$. $\square$

Theorem 19. *Any Borel subgroup of G_n equals B_g for some $g \in G$.*

Proof. Suppose H is a Borel subgroup of G_n. Then H is a connected solvable subgroup of G, and hence, by Theorem 16, it must be of type I or type III. By Proposition 9, it can only be of type I: i.e. conjugate to a subgroup of A or B. In the first case, by Lemma 17(1), it can be conjugated to a subgroup of $\mathrm{SL}_2(\mathbb{C})$. In fact, since H is connected and solvable, it can be conjugated to a subgroup of upper triangular matrices: $U_0 = U \,\cap\, \mathrm{SL}_2(\mathbb{C}) \subseteq U$. In particular, there is $g \in G$ such that $H \subseteq g^{-1} U\, g \,\cap\, G_n$ which is always a proper subgroup of $g^{-1} B\, g \,\cap\, G_n$ This contradicts the maximality of H. Thus H can only be conjugated to a subgroup of B, i.e. there is $g \in G$ such that $H \subseteq g^{-1} B\, g \,\cap\, G_n$. Since H is a maximal solvable subgroup of G_n, we must have $H = g^{-1} B\, g \,\cap\, G_n = B_g$. $\square$

As a consequence of Theorem 19 and Proposition 11(b), we get the following infinite-dimensional generalization of a well-known theorem of Borel [Bo1].

Corollary 8. *Any Borel subgroup of G_n equals its normalizer.*

Now, let $\mathfrak{B}_n$ denote the set of all Borel subgroups of G_n. By Theorem 19, we have a natural inclusion $\iota : \mathfrak{B}_n \hookrightarrow V_n(B)$, which is obviously equivariant with respect to the adjoint action of G_n. Taking quotients by this action and combining the induced map of ι with the inverse of (57), we get

$$\mathfrak{B}_n / \mathrm{Ad}\, G_n \ \hookrightarrow\ \mathcal{C}_n / B \ , \tag{58}$$

which is precisely the embedding (1) mentioned in the Introduction. Our aim now is to prove Theorem 4. We begin by recalling the following important fact proved by G. Wilson in [W, Sect. 6].

Theorem 20 ([W]). *For each n, the variety $\mathcal{C}_n$ has exactly $p(n)$ torus-fixed points (X, Y) which are in bijection with the partitions of n. These points are characterized by the property that both X and Y are nilpotent matrices.*

We will refer to points $(X, Y) \in \mathcal{C}_n$, with X and Y being nilpotent matrices, as 'nilpotent points.' The next observation is an easy consequence of Theorem 20.

Corollary 9. *Let Γ be a subgroup of T containing a cyclic group of order $> n$ (possibly infinite). If a point $(X, Y) \in \mathcal{C}_n$ is fixed by Γ then it is also fixed by T.*

Proof. If (X, Y) is fixed by Γ, then $\mathrm{Tr}(X^k)$ and $\mathrm{Tr}(Y^k)$ vanish for all $k \le n$. Hence $\mathrm{Tr}(X^k) = \mathrm{Tr}(Y^k) = 0$ for all $k > 0$. This means that X and Y are both nilpotent matrices, and the claim follows from Theorem 20. □

Next, for each $(X, Y) \in \mathcal{C}_n$, we define the following canonical map

$$\chi_{(X,Y)} : \ \mathrm{Stab}_B(X, Y) \hookrightarrow B \twoheadrightarrow B/[B, B] \ . \tag{59}$$

Note that the image of (59) depends only on the B-orbit of (X, Y) in $\mathcal{C}_n$ (not on the specific representative). The target of (59) plays the role of an 'abstract' Cartan subgroup of G, which (just as in the finite-dimensional case, *cf.* [CG, Sect. 3.1]) can be identified with a maximal torus:

$$B/[B, B] \cong T \ , \quad [(tx + p(y),\, t^{-1}y + f)] \leftrightarrow (tx,\, t^{-1}y) \ . \tag{60}$$

In terms of (59), we can give the following useful characterization of B-orbits with T-fixed points.

Lemma 18. *A B-orbit of $(X, Y) \in \mathcal{C}_n$ contains a T-fixed point if and only if the map $\chi_{(X,Y)}$ is surjective. For this, it suffices that the image of $\chi_{(X,Y)}$ contains an element of order $> n$.*

Proof. First, in view of (60), it is obvious that $\chi_{(X,Y)}$ is surjective if $T \subseteq \mathrm{Stab}_B(X, Y)$. For the converse, we will prove the existence of a T-fixed point under the assumption that $\chi_{(X,Y)}$ contains an element of order $> n$. By this assumption, there is an element $h = (tx + p(y), t^{-1}y + f) \in \mathrm{Stab}_B\,(X, Y)$ such that t has order $\ge n+1$ in $\mathbb{C}^*$. By the Cayley-Hamilton Theorem, we may assume that $\deg\, p(y) \le n-1$. Applying the automorphism $b_1 := (x - \frac{1}{n}\mathrm{Tr}(X), y - \frac{1}{n}\mathrm{Tr}(Y)) \in B$ to (X, Y), we get a point (X_1, Y_1) with $\mathrm{Tr}(X_1) = \mathrm{Tr}(Y_1) = 0$. Hence $b_1\,[\mathrm{Stab}_B\,(X, Y)]\,b_1^{-1} \subseteq T \ltimes G_y$ and $h_1 = b_1\,h\,b_1^{-1} = (tx + p_1(y), t^{-1}y)$ with $\deg\, p_1(y) \le n-1$. We now show that h_1 can be conjugated to $(tx, t^{-1}y)$. Indeed, write $p_1(y) = \sum_{i=0}^{n-1} a_i y^i$ and conjugate

$$b_2\,h_1\,b_2^{-1} = (tx + tq(y) - q(t^{-1}y) + p_1(y),\, t^{-1}y) \ ,$$

where $b_2 := (x + q(y), y)$ with $q(y) = \sum_{i=0}^{n-1} c_i y^i \in \mathbb{C}[y]$. Setting

$$tq(y) - q(t^{-1}y) + p_1(y) = 0 \ ,$$

we get a linear system for the coefficients of $q(y)$ of the form

$$c_i(t - t^{-i}) = a_i \quad i = 0, \ldots, n-1 \ .$$

Hence, if we take $c_i = a_i\,(t - t^{-i})^{-1}$ for b_2 and set $b := b_2 b_1$, then $b\,h\,b^{-1} = (tx, t^{-1}y)$. Thus $b\,[\mathrm{Stab}_B(X, Y)]\,b^{-1}$ contains $(tx, t^{-1}y)$. By Corollary 9, we now conclude that $b \cdot (X, Y)$ is a nilpotent point, and hence, by Theorem 20, it is T-fixed. □

Now, for $(X,Y) \in \mathcal{C}_n$, let $G_y(X,Y)$ denote the stabilizer of (X,Y) in G_y. Note that $G_y(X,Y) \subseteq \mathrm{Stab}_B(X,Y)$ for any (X,Y), since $G_y \subset B$. The next lemma is a direct consequence of Proposition 13, which is proved in Section 6.6; it shows that all groups $G_y(X,Y)$ are path connected (and hence connected).

Lemma 19. *For any $(X,Y) \in \mathcal{C}_n$, if $(x+q(y),y) \in G_y(X,Y)$, then $(x+\lambda q(y),y) \in G_y(X,Y)$ for any $\lambda \in \mathbb{C}$.*

We now give a classification of B-orbits in $\mathcal{C}_n$ and their isotropy groups.

Proposition 12. *For a B-orbit $\mathcal{O}_B$ in $\mathcal{C}_n$, one and only one of the following possibilities occurs:*

(A) *T acts freely on $\mathcal{O}_B$, $\mathrm{Stab}_B(X,Y) = G_y(X,Y)$ and the map $\chi_{(X,Y)}$ is trivial (i.e., its image is 1) for every $(X,Y) \in \mathcal{O}_B$.*

(B) *$\mathcal{O}_B$ contains a T-fixed (X,Y), $\mathrm{Stab}_B(X,Y) = T \ltimes G_y(X,Y)$, and the map of $\chi_{(X,Y)}$ is surjective.*

(C) *$\mathcal{O}_B$ contains a point (X,Y) such that $\mathrm{Stab}_B(X,Y) = \mathbb{Z}_k \ltimes G_y(X,Y)$ for some $0 < k \leq n$, and the image of $\chi_{(X,Y)}$ is isomorphic to $\mathbb{Z}_k$.*

Proof. Let $\mathcal{O}_B$ be a fixed B-orbit. For any $(X,Y) \in \mathcal{O}_B$, the character map (59) combined with (60) gives the short exact sequence

$$1 \to G_y(X,Y) \to \mathrm{Stab}_B(X,Y) \to K \to 1 \tag{61}$$

where K is the image of $\chi_{(X,Y)}$ in T.

If $K = 1$ for some point in $\mathcal{O}_B$, then $K = 1$ for all $(X,Y) \in \mathcal{O}_B$ and hence $\mathrm{Stab}_B(X,Y) = G_y(X,Y)$ for all $(X,Y) \in \mathcal{O}_B$, which means that T acts freely on $\mathcal{O}_B$. This is case (A).

If K contains an element of order $\geq n+1$ (possibly ∞) for some point in $\mathcal{O}_B$, then, by Lemma 18, $\mathcal{O}_B$ contains a T-fixed point (X,Y) and $K = T$. Then $\mathrm{Stab}_B(X,Y)$ contains T, the above short exact sequence splits, and we have $\mathrm{Stab}_B(X,Y) = T \rtimes G_y(X,Y)$. This is case (B).

Finally, assume that neither (A) nor (B) holds. Then, by Lemma 17, there is still a point $(X,Y) \in \mathcal{O}_B$ such that $\mathrm{Stab}_B(X,Y) \subseteq T \ltimes G_y$. By our assumption, the corresponding $K \subset T$ must be a cyclic group of order k for $0 < k \leq n$. Let $(\lambda, \lambda^{-1}) \in K$ be the generator of K. Write $\phi = (\lambda x + p(y), \lambda^{-1} y)$ for the preimage of (λ, λ^{-1}) in $\mathrm{Stab}_B(X,Y)$. Iterating ϕ, we get

$$\phi^k = (x + \sum_{j=1}^{k} \lambda^{k-j} p(\lambda^{1-j} y), y)$$

Explicitly, if $p(y) = \sum_{i=0}^{m} a_i y^i$, then the coefficient under y^i in the first component of ϕ^k is equal to

$$a_i \sum_{j=1}^{k} \lambda^{-k-j+(1-j)i} = a_i \lambda^i \sum_{j=1}^{k} \lambda^{-j(i+1)}$$

Since $\sum_{j=1}^{k} \lambda^{k-j} = 0$, all these coefficients vanish except those with $i \equiv -1 \pmod{k}$. Thus $\phi^k = (x + k\, p_1(y), y)$, where $p_1(y) = a_{k-1} y^{k-1} + a_{2k-1} y^{2k-1} + \dots$ is a polynomial obtained from $p(y)$ by removing all coefficients except those with $i \equiv -1 \pmod{k}$. Since $\phi^k \in \mathrm{Stab}_B(X,Y)$, by Lemma 19, $(x - \lambda^{-1} p_1(y), y) \in \mathrm{Stab}_B(X,Y)$. Hence $\phi_1 := (\lambda x + p(y) - p_1(y), \lambda^{-1} y) \in \mathrm{Stab}_B(X,Y)$ and $\phi_1^k = 1$. Now, the mapping $(\lambda, \lambda^{-1}) \mapsto \phi_1$ splits (61). Hence $\mathrm{Stab}_B(X,Y) = \mathbb{Z}_k \ltimes G_y(X,Y)$, where $\mathbb{Z}_k$ is generated by ϕ_1. This is case (C). $\square$

We are now ready to prove Theorem 4 from the Introduction.

Proof of Theorem 4. By Theorem 19, any Borel subgroup of G_n has the form $B_g := g^{-1}B\,g \cap G_n$, while $B_g = g^{-1}[\mathrm{Stab}_B\,(X,Y)]\,g$, where $(X,Y) = g\cdot(X_0,Y_0) \in \mathcal{C}_n$. Now, by classification of Proposition 12, the group $\mathrm{Stab}_B(X,Y)$ is connected if and only if the corresponding B-orbit is of type (A) or type (B). Indeed, in case (A), we have $\mathrm{Stab}_B(X,Y) = G_y(X,Y)$. Hence, by Lemma 19, $\mathrm{Stab}_B(X,Y)$ is path connected and therefore connected. Note also that $\mathrm{Stab}_B(X,Y)$ is abelian, since so is $G_y(X,Y)$.

In case (B), we may assume that $\mathrm{Stab}_B(X,Y) = T \ltimes G_y(X,Y)$. Then any element of $\mathrm{Stab}_B(X,Y)$ can be written in the form $b = (ax + q(y), a^{-1}y)$, where $q(y) \in \mathbb{C}[y]$. By Lemma 19, if $b \in \mathrm{Stab}_B(X,Y)$ then $b_t := (ax + t\,q(y), a^{-1}y) \in \mathrm{Stab}_B(X,Y)$ for all $t \in \mathbb{C}$, hence we can join $b = b_1$ to $b_0 = (ax, a^{-1}y) \in T$ within $\mathrm{Stab}_B(X,Y)$. It follows that $\mathrm{Stab}_B(X,Y)$ is connected since so is T. Note that in this case, $\mathrm{Stab}_B(X,Y)$ is a solvable but non-abelian subgroup of G.

In case (C), the group $\mathrm{Stab}_B(X,Y)$ is obviously disconnected. Hence the corresponding B_g cannot be a Borel subgroup of G_n. □

6.5. **Conjugacy classes of non-abelian Borel subgroups.** Following [W], we denote the T-fixed points of $\mathcal{C}_n$ by (X_μ, Y_μ), where $\mu = (n_1, n_2, \ldots, n_k)$ is a partition of n with $n_1 \le n_2 \le \ldots \le n_k$. We consider the B-orbits of these points in $\mathcal{C}_n$ as vertices of the graph Γ_n defined in Section 4.2. For a fixed collection of elements $g_\mu \in G$ such that $g_\mu(X_0,Y_0) = (X_\mu, Y_\mu)$, we define the subgroups $B_\mu \subset G_n$ by

$$B_\mu \,:=\, g_\mu^{-1}\,B\,g_\mu \cap G_n\,. \tag{62}$$

These are B-vertex groups attached to the B-orbits $B(X_\mu,Y_\mu)$ in Γ_n. Geometrically, B_μ are the conjugates of subgroups of B fixing the points (X_μ,Y_μ) in $\mathcal{C}_n$. More explicitly $B_\mu = g_\mu^{-1}\,B(\mu)\,g_\mu$, where $B(\mu) := \mathrm{Stab}_B\,(X_\mu,Y_\mu)$.

As an immediate consequence of Theorem 4, we have

Corollary 10. *B_μ is a Borel subgroup of G_n.*

Next, we prove

Theorem 21. *Any non-abelian Borel subgroup of G_n is conjugate to some B_μ.*

Proof. Suppose H is a non-abelian Borel subgroup of G_n. By Theorem 19, any Borel group is equal to $H = B_g$ for some $g \in G$. Then, by Theorem 4, H is Borel if either (A) T acts freely on corresponding B-orbit or (B) T has a fixed point on the corresponding B-orbit. In the first case, H must be abelian, which contradicts our assumption. In the second case, H is conjugate to $T \ltimes G_y(X,Y)$, where (X,Y) is a nilpotent point. Hence H is conjugate to B_μ for some μ. □

Lemma 20. *B_μ contains no proper subgroup of finite index.*

Proof. Similar to the proof of Lemma 14. □

Now we are ready to prove Steinberg's Theorem in full generality.

Proof of Theorem 6. ($\Rightarrow$) Let H be a Borel subgroup of G_n. Then, by Theorem 21, H is conjugate to B_μ. Hence, by Proposition 11(c) and Lemma 20, H satisfies properties (B1) and (B2) respectively.

($\Leftarrow$) Let H be a subgroup of G_n satisfying (B1) and (B2). By Theorem 16, it is then either of type I or type III. By Proposition 9, it cannot be of type III.

Therefore, it is conjugate to either a subgroup of A or a subgroup of B. Suppose that it is conjugate to a subgroup of A. The image of $g^{-1}Hg \to A \to \mathrm{SL}_2(\mathbb{C})$ is then a solvable subgroup of $\mathrm{SL}_2(\mathbb{C})$. We denote this group by S. By Theorem 17, S has a finite index normal subgroup T, which is a subgroup of upper triangular matrices in $\mathrm{SL}_2(\mathbb{C})$. By Lemma 13(b), the group S, being a homomorphic image of H, contains no proper subgroup of finite index. Thus $S = T$ and H is conjugate to a subgroup of upper triangular matrices: $U_0 = U \cap \mathrm{SL}_2(\mathbb{C}) \subseteq U$. In particular, there is $g \in G$ such that $H \subseteq g^{-1}\, U\, g \cap G_n$ which is always a proper subgroup of $g^{-1}\, B\, g \cap G_n$. This contradicts property (B1). Hence H can only be conjugate to a subgroup of B. Thus $H \leq g^{-1}\, B\, g \cap G_n$ for some $g \in G$. Since H is maximal solvable, we have $H = g^{-1}\, B\, g \cap G_n$, thus $H = g^{-1}\,\mathrm{Stab}_B(X,Y)\, g$. Since H is non-abelian, by Proposition 12, the group $\mathrm{Stab}_B(X,Y)$ is either (1) $T \ltimes G_y(X,Y)$ or (2) $\mathbb{Z}_k \ltimes G_y(X,Y)$. By assumption (B2), H does not contain a subgroup of finite index, hence (2) is impossible. Therefore we must have $\mathrm{Stab}_B(X,Y) = T \ltimes G_y(X,Y)$ and hence H is conjugate to some B_μ. □

We will prove that the subgroups B_μ are pairwise non-conjugate in G_n. We begin with the following lemma, the proof of which is essentially contained in [W]. For reader's convenience, we provide full details.

Lemma 21. *The nilpotent points (X_μ, Y_μ) in $\mathcal{C}_n$ belong to distinct B-orbits.*

Proof. Consider the subgroup B_0 consisting of the automorphisms $(x+p(y), y) \in G$ with $p(0) = 0$. It is easy to see that any two nilpotent points are in the same B-orbit iff they are in the same B_0-orbit. Indeed, T fixes each of the nilpotent points, hence does not contribute to the B-orbit. On the other hand, applying an automorphism with nonzero constant terms to a nilpotent point moves it to a point with a nonzero trace, which is not nilpotent. Therefore we will only consider orbits of B_0. By [W, Proposition 6.11], the points (X_μ, Y_μ) are exactly the centers of distinct n-dimensional cells in $\mathcal{C}_n$ which have pairwise empty intersection. Now, if we show that these cells contain the B_0-orbits of (X_μ, Y_μ), the result will follow. We start by looking at the simplest case the point corresponding with partition: $\mu = \mu(n,r)$ where $\mu(n,r) = (1, \ldots, 1, n-r+1)$. In this case (X_μ, Y_μ) is given by

$$X_\mu = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & a_{n-1} & 0 \end{pmatrix} \quad , \quad Y_\mu = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

where $(a_1, \ldots, a_{n-1}) = (1, 2, \ldots, r-1; -(n-r), \ldots, -2, -1)$. Then, the B_0-orbit of (X_μ, Y_μ) consists of the points (X, Y_μ), where $X = X_\mu + \sum_{k=1}^{n-1} X^{(k)}$ with matrices $X^{(k)}$ having nonzero terms only on the k-th diagonal. Applying a transformation Q_t, which is essentially a scaling transformation followed by conjugation by $\mathrm{diag}(1, t, \ldots, t^{n-1})$ (see [W, (6.5)]), we obtain

$$Q_t(X, Y_\mu) = (X_\mu + \sum_{k=1}^{n-1} t^{-k-1} X^{(k)}, Y_\mu) \tag{63}$$

As $t \to \infty$, we see that $Q_t(X, Y_\mu) \to (X_\mu, Y_\mu)$, hence (X, Y_μ) is still in a cell with the center (X_μ, Y_μ).

More generally, consider the partition $\mu = \mu(n_1, r_1, \ldots n_k, r_k)$ which corresponds to the Young diagram with one-hook partitions $(1, \ldots, 1, n_i - r_i + 1)$ placed inside each other; such that neither the arm nor the leg of any hook is allowed to poke out beyond the preceding one. In this case, $Y_\mu = \oplus_{i=1}^{k} J(n_i)$ as a sum of several nilpotent Jordan blocks of dimensions n_k; and X_μ is a block matrix consisting of the diagonal blocks $X_{ii} = X_{(1,\ldots,1,n_i-r_i+1)}$ described as in the previous paragraph and certain (unique) matrices X_{ij} with non-zero entries only on the $(r_j - r_i - 1)$-th diagonal. The B_0-orbit of (X_μ, Y_μ) then consists of the points $(\tilde{X}, Y_\mu)$, where $\tilde{X}_{ij} = X_{ij}$ for $i \neq j$ and $\tilde{X}_{ii} = X_{ii} + \sum_{k=1}^{n_i-1} X^{(i,k)}$ is the sum of matrices $X^{(i,k)}$ with only nonzero terms on the kth diagonal of the corresponding block matrix. Once again, looking at $Q_t(\tilde{X}, Y_\mu)$ one can easily show that the diagonal blocks $\tilde{X}_{ii}$ flow to X_{ii} as $t \to \infty$. On the other hand, the only non-zero diagonal of $\tilde{X}_{ij}$ is the $(r_j - r_i - 1)$-th diagonal, counting within the (i,j)-block; or, if we count diagonals inside the big matrix $\tilde{X}$, it is the one with number $q_j - q_i - 1$, where

$$q_i := n_1 + \ldots + n_{i-1} + r_i \ .$$

Thus, the map Q_t multiplies the non-zero diagonal of $\tilde{X}_{ij}$ by $t^{q_i - q_j}$. If we now conjugate by the block-scalar matrix $\oplus\, t^{-q_i} I_{n_i}$, then the (i,j)-block gets multiplies by $t^{q_j - q_i}$, so we get X_{ij}. Thus, summing up, we obtain that $Q_t(\tilde{X}, Y_\mu) \to (X_\mu, Y_\mu)$ as $t \to \infty$, hence the corresponding B_0-orbit is in the cell. □

Theorem 22. *The subgroups B_μ are pairwise non-conjugate in G_n, i.e. there is no $g \in G_n$ such that $g^{-1} B_\mu g = B_\lambda$ unless $\mu = \lambda$.*

Proof. This is a consequence of Lemma 16 (see (57)) and Lemma 21. □

Now, we can prove Theorem 5 and Corollary 1 stated in the Introduction.

Proof of Theorem 5. Combine Theorem 21 and Theorem 22. □

Proof of Corollary 1. Suppose that there exists an (abstract) group isomorphism $G_k \cong G_n$ for some k and n. Then, by Theorem 6, it must induce a bijection between the sets of conjugacy classes of non-abelian Borel subgroups in G_k and G_n. By Theorem 5, these sets are finite sets consisting of $p(k)$ and $p(n)$ elements. Hence $p(k) = p(n)$ and therefore $k = n$. □

6.6. **Adelic construction of Borel subgroups.** We conclude this section by giving an explicit description of the special subgroups $B(\mu)$. To this end we will use an infinite-dimensional *adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad}}$ introduced in [W1]. We recall that $\mathrm{Gr}^{\mathrm{ad}}$ is the space parametrizing all primary decomposable subspaces of $\mathbb{C}[z]$ modulo rational equivalence. To be precise, a subspace $W \subseteq \mathbb{C}[z]$ is called primary decomposable if there is a finite collection of points $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \subset \mathbb{C}$ such that $W = \bigcap_{i=1}^{N} W_{\lambda_i}$, where W_λ is a λ-primary (i.e., containing a power of the maximal ideal $\mathfrak{m}_\lambda$) subspace of $\mathbb{C}[z]$. Two such subspaces, say W and W', are (rationally) equivalent if $pW = qW'$ for some polynomials p and q. Every equivalence class $[W] \in \mathrm{Gr}^{\mathrm{ad}}$ contains a unique *irreducible* subspace, which is characterized by the property that it is not contained in a proper ideal of $\mathbb{C}[z]$. We may therefore identify $\mathrm{Gr}^{\mathrm{ad}}$ with the set of irreducible primary decomposable subspaces in $\mathbb{C}[z]$.

Now, by [W] and [BW], there is a natural bijection $\beta : \bigsqcup_{n \geq 0} \mathcal{C}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathrm{ad}}$, which is equivariant under G. It is not easy to describe the action of the full group G on $\mathrm{Gr}^{\mathrm{ad}}$; however, for our purposes, it will suffice to know the action of its subgroup

G_y, which is not difficult to describe. We will use the construction of the action of G_y on $\mathrm{Gr}^{\mathrm{ad}}$ given in [BW] (where G_y is denoted by Γ).

Let $\mathcal{H}$ denote the space of entire analytic functions on $\mathbb{C}$ equipped with its usual topology (uniform convergence on compact subsets). Given a subspace $W \subseteq \mathbb{C}[z]$ we write $\overline{W} \subseteq \mathcal{H}$ for its completion in $\mathcal{H}$, and conversely, given a closed subspace $\mathcal{W} \subseteq \mathcal{H}$ we set $\mathcal{W}^{\mathrm{alg}} := \mathcal{W} \cap \mathbb{C}[z]$. Then, for any $q \in \mathbb{C}[z]$, we define

$$e^q \cdot W := (e^q\, \overline{W})^{\mathrm{alg}}$$

The action of G_y under β transfers to $\mathrm{Gr}^{\mathrm{ad}}$ as follows (see [BW, Sect. 10]): if $W = \beta(X, Y) \in \mathrm{Gr}^{\mathrm{ad}}$ then

$$e^q \cdot W = \beta(X + q'(Y), Y)\ , \quad \forall\, q \in \mathbb{C}[z]\ .$$

Now, for any $W \in \mathrm{Gr}^{\mathrm{ad}}$, put

$$A_W := \{q \in \mathbb{C}[z]\ :\ qW \subseteq W\}\,.$$

Clearly A_W is a commutative algebra, W being a finite module over A_W. Geometrically, A_W is the coordinate ring of a rational curve $X = \mathrm{Spec}(A_W)$, on which W defines a (maximal) rank 1 torsion-free coherent sheaf $\mathfrak{L}$. The inclusion $A_W \hookrightarrow \mathbb{C}[z]$ gives normalization $\pi : \mathbb{A}^1 \to X$ (which is set-theoretically a bijective map). In this way, $\mathrm{Gr}^{\mathrm{ad}}$ parametrizes the isomorphism classes of triples $(\pi, X, \mathfrak{L})$ (see [W1]).

Proposition 13. *For any* $W \in \mathrm{Gr}^{\mathrm{ad}}$*,* $\mathrm{Stab}_{G_y}(W) = \{(x+q'(y), y) \in G\ :\ q \in A_W\}$*.*

Proof. By [BW, Lemma 2.1] and the above discussion, the claim is equivalent to

$$A_W\ =\ \{q \in \mathbb{C}[z]\ :\ e^q\, \overline{W} = \overline{W}\}\ .$$

The inclusion '$\subset$' is easy: if $q \in A_W$ then $q^n\, W \subset W$ for all $n \in \mathbb{N}$, hence $e^q\, W \subset \overline{W}$ and therefore $e^q\, \overline{W} = \overline{W}$.

To prove the other inclusion it is convenient to use the 'dual' description of $\mathrm{Gr}^{\mathrm{ad}}$ in terms of algebraic distributions (see [W1]). To this end assume that W is supported on $\{\lambda_1, \lambda_2, \dots, \lambda_N\} \subset \mathbb{C}$. Then, for each $\lambda_i \in \mathrm{supp}(W)$, there is a finite-dimensional subspace $W^*_{\lambda_i}$ of linear functionals on $\mathcal{H}$ supported at λ_i such that[9]

$$W = \{f \in \mathbb{C}[z]\ :\ \langle \varphi_i,\, f\rangle = 0 \text{ for all } \varphi_i \in W^*_{\lambda_i} \text{ and for all } i = 1, 2, \dots, N\}\ .$$

By [BW, Lemma 2.1], we then also have

$$\overline{W} = \{f \in \mathcal{H}\ :\ \langle \varphi_i,\, f\rangle = 0 \text{ for all } \varphi_i \in W^*_{\lambda_i} \text{ and for all } i = 1, 2, \dots, N\}\ .$$

Now, suppose that $e^q\, \overline{W} = \overline{W}$ for some $q \in \mathbb{C}[z]$. Then $e^{tq}\, \overline{W} = \overline{W}$ for all $t \in \mathbb{C}$. Indeed, for fixed $\varphi_i \in W^*_{\lambda_i}$ and $f \in \overline{W}$, the function $P(t) := \langle \varphi_i,\, e^{tq} f\rangle$ is obviously a quasi-polynomial in t of the form $P(t) = p(t)\, e^{q(\lambda_i)t}$, where $p(t) \in \mathbb{C}[t]$. Since $e^q\, \overline{W} = \overline{W}$ implies $e^{kq}\, \overline{W} = \overline{W}$ for all $k \in \mathbb{Z}$, we have $P(k) = 0$ and hence $p(k) = 0$ for all $k \in \mathbb{Z}$. This implies $P(t) \equiv 0$. In particular, we have $P'(0) = \langle \varphi_i,\, qf\rangle = 0$. Since this equality holds for all $\varphi \in W^*_{\lambda_i}$, for all i and for all $f \in W$, we conclude $qW \subseteq W$. Thus $q \in A_W$. $\square$

[9]Note that the elements of $W^*_{\lambda_i}$ can be written as $\varphi_i = \sum_k c_{ik}\, \delta^{(k)}(z - \lambda_i)$, where $\delta^{(k)}(z - \lambda_i)$ are the derivatives of the δ-function with support at λ_i.

Now, let (X_μ, Y_μ) be the T-fixed point of $\mathcal{C}_n$ corresponding to a partition $\mu = \{n_1 \le n_2 \le \ldots \le n_k\}$. Then, the corresponding (irreducible) primary decomposable subspace of $\mathrm{Gr}^{\mathrm{ad}}$ is given by

$$W_\mu = \mathrm{span}\{1,\, x^{r_1},\, x^{r_2},\, x^{r_3},\, \ldots\}\ ,$$

where $r_i = i + n_k - n_{k-i}$ (with convention $n_j = 0$ for $j < 0$). Write $R_\mu := \{r_0 = 1,\, r_1,\, r_2,\, \ldots\}$ for the set of exponents of monomials occurring in W_μ, and denote by $S_\mu := \{k \in \mathbb{N} \,:\, k + R_\mu \subset R_\mu\}$ the subsemigroup of $\mathbb{N}$ preserving R_μ. Then $A_{W_\mu} = \mathrm{span}\{x^s \,:\, s \in S_\mu\}$, and as a consequence of Proposition 13, we get

Corollary 11. *For any partition μ, $B(\mu) = T \ltimes G_{\mu,y}$, where $G_{\mu,y}$ is the subgroup of G_y generated by the transformations $\{(x + \lambda y^{s-1}, y) \,:\, s \in S_\mu\,,\, \lambda \in \mathbb{C}\}$.*

To illustrate Corollary 11, we list below all special Borel subgroups of G_n for $n = 1,\, 2,\, 3,\, 4$.

6.6.1. *Examples.* For $n = 1$, there is only one T-fixed point $(0, 0) \in \mathcal{C}_1$ and the corresponding Borel subgroup is

$$B_{(1)} = T \ltimes \{\Psi_{cy^k} \,|\, c \in \mathbb{C}, k \ge 1\} = \{(ax + cy^k, a^{-1}y) \mid a \in \mathbb{C}^*, c \in \mathbb{C}, k \ge 1\}\,.$$

For $n = 2$, the fixed points are $(X_{(2)}, Y_{(2)})$ and $(X_{(1,1)}, Y_{(1,1)})$, where

$$X_{(2)} := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad , \quad X_{(1,1)} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad Y_{(2)} = Y_{(1,1)} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ .$$

The corresponding Borel subgroups are given by

$$\begin{aligned} B_{(2)} &= T \ltimes \{\Psi_{cy^k} \,|\, c \in \mathbb{C}, k \ge 2\}\,, \\ B_{(1,1)} &= T \ltimes \{\Phi_{cx^k} \,|\, c \in \mathbb{C}, k \ge 2\} \end{aligned}$$

For $n = 3$, the fixed points are

$$X_{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \ , \quad X_{(1,1,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \ , \quad X_{(1,2)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

and

$$Y_{(3)} = Y_{(1,1,1)} = Y_{(1,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The corresponding Borel subgroups are given by

$$\begin{aligned} B_{(3)} &= T \ltimes \{\Psi_{cy^k} \,|\, c \in \mathbb{C}, k \ge 3\}\,, \\ B_{(1,1,1)} &= T \ltimes \{\Phi_{cx^k} \,|\, c \in \mathbb{C}, k \ge 3\}\ , \\ B_{(1,2)} &= \Psi_{-y^2}\ \Phi_{-\frac{x^2}{2}}\ \Psi_{-2y^2}\, B(1,2)\, \Psi_{2y^2}\, \Phi_{\frac{x^2}{2}}\ \Psi_{y^2}\ , \end{aligned}$$

where

$$B(1,2) := T \ltimes \{\Psi_{q(y)} \,|\, q(y) \in \mathbb{C}y + y^3\mathbb{C}[y]\}$$

For $n = 4$, there are five fixed points:

$$X_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \quad X_{(1,3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \quad X_{(1,1,2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$X_{(2,2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -3 & 0 & 0 & 0 \end{pmatrix} , \quad X_{(1,1,1,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$Y_{(2,2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad Y_{\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

where $\mu = \{(4),\, (1,3),\, (1,1,2),\, (1,1,1,1)\}$. The corresponding Borel subgroups are

$$\begin{aligned} B_{(4)} &= B(4)\,, \quad B_{(1,3)} = \Psi_{-y^3}\,\Phi_{\frac{x^3}{6}}\,\Psi_{-3y^3}\,B(1,3)\,\Psi_{3y^3}\,\Phi_{-\frac{x^3}{6}}\,\Psi_{y^3}\ , \\ B_{(1,1,2)} &= \Psi_{-y^3}\,\Phi_{\frac{x^3}{3}}\,\Psi_{3y^3}\,B(1,1,2)\,\Psi_{-3y^3}\,\Phi_{-\frac{x^3}{3}}\,\Psi_{y^3} \\ B_{(1,1,1,1)} &= \Psi_{-y^3}\,\Phi_{\frac{x^3}{2}}\,\Psi_{-y^3}\,B(1,1,1,1)\,\Psi_{y^3}\,\Phi_{-\frac{x^3}{2}}\,\Psi_{y^3} \\ B_{(2,2)} &= \Psi_{-y^2}\,\Phi_{\frac{-x^2}{4}}\,\Psi_{-2y^2}\,B(2,2)\,\Psi_{2y^2}\,\Phi_{\frac{x^2}{4}}\,\Psi_{y^2}\,, \end{aligned}$$

where

$$\begin{aligned} B(4) &= T \ltimes \{\Psi_{q(y)}\,|\,q(y) \in y^4\mathbb{C}[y]\}\,, \\ B(1,3) &= T \ltimes \{\Psi_{q(y)}\,|\,q(y) \in \mathbb{C}y^2 + y^4\mathbb{C}[y]\}\ , \\ B(1,1,2) &= T \ltimes \{\Psi_{q(y)}\,|\,q(y) \in \mathbb{C}y^2 + y^4\mathbb{C}[y]\}\ , \\ B(1,1,1,1) &= T \ltimes \{\Psi_{q(y)}\,|\,q(y) \in y^4\mathbb{C}[y]\}\ , \\ B(2,2) &= T \ltimes \{\Psi_{q(y)}\,|\,q(y) \in y^3\mathbb{C}[y]\}\ . \end{aligned}$$

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